7. (14 points) Prove that the sum of the fifth powers of any five consecutive nonnegative integers is divisible by 25.

We prove that 25 divides $n^5+(n+1)^5+(n+2)^5+(n+3)^5+(n+4)^5$ for all $0 \le n$. The proof is by induction on n. When $n=0$, we have $n^5+(n+1)^5+(n+2)^5+(n+3)^5+(n+4)^5 = 0^5+1^5+2^5+3^5+4^5 = 1300$, which is divisible by 25. By induction we assume that $n^5+(n+1)^5+(n+2)^5+(n+3)^5+(n+4)^5$ is divisible by 25 for some n. We must show that $(n+1)^5+(n+2)^5+(n+3)^5+(n+4)^5+(n+5)^5$ is divisible by 25. Thus, it suffices to show that $(n+5)^5-n^5$ is divisible by 25 but $(n+5)^5-n^5 = 25n^4+250n^3+1250n^2+3125n+3125$, which is divisible by 25.

8. (14 points) Write

$$\binom{300}{300}+\binom{301}{300}+\binom{302}{300}+\binom{303}{300}+\cdots+\binom{2997}{300}+\binom{2998}{300}+\binom{2999}{300}+\binom{3000}{300}$$

as one binomial coefficient. Explain your work.

This sum is equal to

$$\binom{300}{0}+\binom{301}{1}+\binom{302}{2}+\binom{303}{3}+\cdots+\binom{2997}{2697}+\binom{2998}{2698}+\binom{2999}{2699}+\binom{3000}{2700}.$$

Apply the empty waffle identity

$$\sum_{k=0}^{p}\binom{k+f-1}{k}=\binom{p+f}{p}$$

with $f=301$, $p=2700$ to get $\binom{3001}{2700} = \binom{3001}{301}$

garbage but ans → 8
plausible but wrong → 5