Math 574, Exam 2, Solutions, Spring 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 10 problems. Each problem is worth 5 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website a few hours after the exam is finished.

1. **Let** $S = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$ **and let** $f\colon S \to S$ **be an onto function. Does** f **have to be one-to-one? Prove or give a counter-example.**

OF COURSE. I am supposed to prove that if f is onto, then f is one-to-one. I prove the contrapositive of the given statement. That is, I prove that if f is not one-to-one, then f is not onto.

Suppose f is not one-to-one. Then there are two elements $s \neq s'$ in S with $f(s) = f(s')$. The domain of S consists of 15 elements. Two of these elements are sent to the same place. So the image of f contains AT MOST 14 elements. The target of f consists of 15 elements. The target of f has at least one more element than the image of f. Thus, there is at least one $s'' \in S$ with s'' not in the image of S. We conclude that f is not onto.

2. **Let** S **be the set of positive integers and let** $f\colon S \to S$ **be an onto function. Does** f **have to be one-to-one? Prove or give a counter-example.**

OF COURSE NOT!!! Define $f(n)$ to be the greatest integer less than or equal to $\frac{n+1}{2}$. (In other words, $f(1) = f(2) = 1$, $f(3) = f(4) = 2$, $f(5) = f(6) = 3$, etc.) We see that f is onto but not one-to-one.

3. **Recall that the Fibonacci numbers are:** $f_1 = 1$**,** $f_2 = 1$**, and for** $n \geq 3$ $f_n = f_{n-1} + f_{n-2}$**. Prove that** $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ **whenever** n **is a positive integer.**

We prove this result by induction on n.

Base case: When $n = 1$, the left side is $f_1 = 1$ and the right side is $f_2 = 1$. We have equality.

Inductive Hypothesis: Fix a positive integer n. Assume that

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}.$$

We will prove that: $f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} = f_{2n+2}$. The inductive hypothesis ensures that

$$f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} = (f_1 + f_3 + \cdots + f_{2n-1}) + f_{2n+1} = f_{2n} + f_{2n+1}.$$

The definition of the Fibonacci numbers tells us that $f_{2n} + f_{2n+1} = f_{2n+2}$. We have completed the proof of the inductive step; and therefore we have completed the proof the result.

4. **Let S, T, and U be sets, and let $f\colon S \to T$ and $g\colon T \to U$ be functions. Suppose that $g \circ f$ is onto. For each question, prove or give a counterexample.**
   (a) Does f have to be onto?
   (b) Does g have to be onto?

   (a) The function f does NOT have to be onto. Consider $S = U = \{1\}$, $T = \{1, 2\}$, $f(1) = 1$, $g(1) = g(2) = 1$. We see that $g \circ f$ is onto, but f is not onto.

   (b) The function g DOES have to be onto. Let u be an arbitrary element of U. The function $g \circ f$ is onto; so, there exists an element $s \in S$ with $g \circ f(s) = u$. Thus $f(s)$ is an element of T and g sends THIS element of T to u.

5. **What is a closed formula for $\sum\limits_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$? Prove your answer. (Recall that a closed formula does not have any summation signs or any dots.)**

We prove by induction that $\sum\limits_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

Base case: We see that when $n = 1$, then $\sum\limits_{k=1}^{n} k^3$ and $\frac{n^2(n+1)^2}{4}$ are both equal to 1.

Induction Hypothesis: Fix an integer n with $1 \le n$. Assume

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

We will prove that $\sum\limits_{k=1}^{n+1} k^3 = \frac{(n+1)^2(n+2)^2}{4}$. The left side is equal to

$$\sum_{k=1}^{n} k^3 + (n+1)^3.$$

We apply the induction hypothesis to see that the left side is equal to

$$\frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2}{4}[n^2+4(n+1)] = \frac{(n+1)^2}{4}[n^2+4n+4]$$

$$= \frac{(n+1)^2}{4}(n+2)^2,$$

and this is the right side. We have completed the proof of the inductive step; and therefore, we have completed the proof of the result.

6. **Goldbach's conjecture states that every even integer greater than 2 is the sum of two primes. Prove that Goldbach's conjecture is equivalent to the statement that every integer greater than 5 is the sum of three primes.**

Assume the original conjecture. Prove the alternate form. Let n be an integer greater than 5. If n is even, then $n-2$ is an even integer greater than 2 and Goldbach's conjecture ensures that there exist prime numbers p and q with $p+q=n-2$. Thus, $p+q+2=n$ and the conclusion of the alternate form holds for n. If n is odd, then $n-3$ is an even integer greater than 2. Once again Goldbach's conjecture ensures that there exist prime numbers p and q with $p+q=n-3$. Thus, $p+q+3=n$. In any event, n is the sum of three primes.

Assume the alternate form. Prove the original conjecture. Let $n>2$ be an even integer. We see that $n+2$ is an arbitrary integer greater than 5. The alternate form of the conjecture ensures that there exist prime numbers p, q, and r with $n+2=p+q+r$. We notice that at least one of the numbers p, q, and r must be even (because three odd numbers add up to an odd number and $n+2$ is even). The only even prime number is 2. So one of the three prime numbers p, q or r is equal to 2. Re-label, if necessary, in order to have $r=2$. We now subtract 2 from each side of $n+2=p+q+2$ to see that $n=p+q$.

7. **Prove that every integer greater than 11 is the sum of 2 composite numbers.**

Let n be an integer greater than 11. We notice that $n-4$, $n-6$, and $n-8$ all are integers greater than 3. Notice that 3 must divide one of these three integers.

Indeed, there are only three possibilities for the remainder after $n-8$ is divided by 3. If the remainder is 0, then 3 divides $n-8$. If the remainder is 1, then 3 divides $n-6$. If the remainder is 2, then 3 divides $n-4$. Thus, one of the three numbers $n-8$, $n-6$, and $n-4$ are composite. On the other hand, 4, 6, and 8 all are composite; so, $n=(n-4)+4=(n-6)+6=(n-8)+8$ is the sum of two composite numbers.

8. **For each positive integer n, let S_n be the following set of real numbers:**

$$S_n = \{x \in \mathbb{R} \mid \tfrac{1}{n} \le x < 2 + \tfrac{1}{n}\}.$$

What is $\bigcup_{n=1}^{75} S_n$? What is $\bigcap_{n=1}^{75} S_n$? I only want the answer. I do not need to see any work.

We see that

$$\boxed{\bigcup_{n=1}^{75} S_n = \{x \in \mathbb{R} \mid \tfrac{1}{75} \le x < 3 \quad \text{and} \quad \bigcap_{n=1}^{75} S_n = \{x \in \mathbb{R} \mid 1 \le x < 2 + \tfrac{1}{75}\}.}$$

9. **Let S be a set of $n+1$ integers between 1 and $2n$. Prove that at least one integer from S divides another integer from S.**

We will prove the statement by induction on n.

Base case: If $n=1$, then S consists of two numbers from $\{1,2\}$; so, $S=\{1,2\}$ and one of the integers from S (namely 1) does indeed the other integer from S (namely 2).

Inductive step: Let n be some fixed integer with $2 \le n$. We suppose that every set T of n integers between 1 and $2n-2$ contains an integer which divides another integer from the set T.

Let S be a set of $n+1$ integers between 1 and $2n$. **We will prove** that at least one integer from S divides another integer from S.

We give names to the elements of S: $s_1 < s_2 < \cdots < s_{n+1}$. There are three cases.

Case 1: $s_{n+1} < 2n$**.** In this case, $s_n < 2n-2$ (since $s_n < s_{n+1} \le 2n-1$). We may apply the induction hypothesis to the set $T = S \setminus \{s_{n+1}\}$. We notice that T is a set of n integers between 1 and $2n-2$. The induction hypothesis guarantees that some element of T divides some other element of T. But T sits inside S; so, some element of S divides some other element of S.

Case 2: $s_{n+1} = 2n$ and some s_i divides s_{n+1} for some $i < n+1$. There is nothing for us to prove in this case; since, in this case, one element of S (namely s_i) divides another element of S (namely s_{n+1}).

Case 3: $s_{n+1} = 2n$ and no s_i divides s_{n+1} for any $i < n+1$. First notice that n is not an element of S because n divides $2n$, but none of the elements of S (except s_{n+1}) divide $2n$. Let T be the set $\{n\} \cup S \setminus \{s_n, s_{n+1}\}$. Observe that the induction hypothesis applies to T. Indeed, T consists of n integers between 1 and $2n-2$. (We know that $s_{n-1} < s_n < s_{n+1} = 2n$.) The induction hypothesis guarantees that some element of T divides some other element of T. We are not quite finished yet, because T contains n, which is not in S. We have to make sure that the division $t_i | t_j$, for some $t_i \neq t_j$ in T, which is guaranteed by the induction hypothesis, involves the elements of $S \cap T$ and not n. But this is easy. We must rule out n as the element of t_i of T and also as the element t_j of T.

We make sure that n does not divide any element of T (other than n itself): Every element that of T is less than $2n$, and n does not divide any integers between 1 and $2n-1$, except n.

We make sure that none of the elements of T (other than n) divide n. Our hypothesis for case 3 says that none of the elements of T (other than n) divide $2n$; and therefore, none of the elements of T (other than n) divide n.

The proof of case 3 is now complete. The induction hypothesis guarantees that some element of T divides some other element of T. We know that neither of these elements is n. Every element of T, other than n, is also in S; hence, some element of S divides some other element of S.

In each of the three cases, we have proven that some element of S divides some other element of S. The proof of the inductive step is complete; and therefore the proof of the result is complete.

10. **Prove that for every positive integer n, there does exist a set T of n integers between 1 and $2n$ such that no integer from T divides any other integer from T.**

Let $\boxed{T = \{n+1, n+2, \ldots, 2n\}}$. We see that T consists of n distinct integers between 1 and $2n$; however, no element of T divides any other element of T.