Math 554, Exam 1, Summer 2006

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. **Leave room on the upper left hand corner of each page for the staple.** Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

There are 7 problems. Problem 1 is worth 8 points. Each of the other problems is worth 7 points. The exam is worth a total of 50 points.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website later this afternoon.

1. **State the least upper bound axiom of the real numbers. Use complete sentences. Include everything that is necessary, but nothing more.**

Every non-empty set of real numbers which is bounded from above has a supremum.

2. **Define "limit of a sequence". Use complete sentences. Include everything that is necessary, but nothing more.**

The *limit of the sequence* of real numbers $\{a_n\}$ is the real number p if for all $\varepsilon > 0$, there exists an integer n_0 such that whenever n is an integer with $n > n_0$, then $|a_n - p| < \varepsilon$.

3. **Suppose X and Y are sets, $f\colon X \to Y$ and $g\colon Y \to X$ are functions, and $g \circ f$ is the identity function on X. (In other words, $g(f(x)) = x$ for all $x \in X$.)**
   (a) **Does the function g have to be one-to-one? If yes, prove it. If no, give a counter example. Write in complete sentences.**

NO. Let $X = \{1\}$, $Y = \{1,2\}$, $f(1) = 1$, $g(1) = g(2) = 1$. It is clear that $f\colon X \to Y$ and $g\colon Y \to X$ are functions and that $g(f(x)) = x$ for all x in X.

It is also clear that g is not one-to-one, because 1 and 2 are distinct elements of Y with $g(1) = g(2)$.

(b) **Does the function g have to be onto? If yes, prove it. If no, give a counter example. Write in complete sentences.**

YES. Let x be an arbitrary element of the set X. We know that $f(x)$ is an element of Y and that $g(f(x)) = x$.

4. **Suppose that $\{a_n\}$ is a sequence which converges to a and $\{b_n\}$ is a sequence that converges to b. Prove that $\{a_n + b_n\}$ is a sequence which converges to $a + b$. I expect a complete, coherent argument. Write in complete sentences.**

Fix $\varepsilon > 0$. Observe that the triangle inequality yields that

$$|(a_n + b_n) - (a + b)| \le |a_n - a| + |b_n - b|.$$

The sequence $\{a_n\}$ converges to a, so there exists a natural number n_1 so that for all $n > n_1$, $|a_n - a| < \frac{\varepsilon}{2}$. The sequence $\{b_n\}$ converges to b, so there exists a natural number n_2 so that for all $n > n_2$, $|b_n - b| < \frac{\varepsilon}{2}$. Let $n_0 = \max\{n_1, n_2\}$. If $n > n_0$, then

$$|(a_n + b_n) - (a + b)| \le |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We conclude that the sequence $\{a_n + b_n\}$ converges to $a + b$.

5. **Prove that between any two real numbers there exists an irrational number. Give a complete proof. Write in complete sentences. If you quote some result we did in class, be sure to quote the complete result.**

Let $a < b$ be real numbers. In class we proved that between any two real numbers there is a rational number. We apply the result from class to the real numbers $0 < b - a$ to find a rational number q with $0 < q < b - a$. We notice that $\frac{\sqrt{2}}{2}$ is an irratioanl number with $0 < \frac{\sqrt{2}}{2} < 1$; hence, $0 < \frac{\sqrt{2}}{2}q < q < b - a$. Add a to every part of the inequality to see that $a < \frac{\sqrt{2}}{2}q + a < q + a < b$. If a is an irrational number, then $q + a$ is an irrational number between a and b. If a is a rational number, then $\frac{\sqrt{2}}{2}q + a$ is an irrational number between a and b.

6. **Consider the sequence $\{a_n\}$ with $a_1 = \sqrt{12}$, and $a_n = \sqrt{12 + a_{n-1}}$ for $n \ge 2$. Prove that the sequence $\{a_n\}$ converges. Find the limit of the sequence $\{a_n\}$. Write in complete sentences.**

It is clear that every term a_n is at most 4. We see that $a_1 \le 4$. If $a_{n-1} \le 4$, then $a_{n-1} + 12 \le 16$; so $a_n = \sqrt{a_{n-1} + 12} < \sqrt{16} = 4$. It is also clear that the sequence is an increasing sequence. We just saw that $a_n \le 4$ for all n. Multiply both sides by the positive number $a_n + 3$ to see that $a_n^2 + 3a_n \le 4a_n + 12$. In other words, we have $a_n^2 \le a_n + 12$. The numbers a_n and $a_n + 12$ are both positive. it follows that $a_n \le \sqrt{a_n + 12} = a_{n+1}$. The sequence $\{a_n\}$ is an increasing bounded sequence. We proved in class that every monotone bounded sequence converges. It follows that the sequence $\{a_n\}$ converges. We know that $\lim\limits_{n\to\infty} a_n$ exists. Let L be the name of this limit. Take the limit of both sides of $a_n = \sqrt{12 + a_{n-1}}$ to see that $L = \sqrt{12 + L}$, or $L^2 = 12 + L$, which is $L^2 - L - 12 = 0$. This equation factors to become $(L-4)(L+3) = 0$; hence $L = 4$ or $L = -3$. Every a_n is positive so $L = -3$ is not possible. We conclude that $\boxed{\lim\limits_{n\to\infty} a_n = 4}$.

7. **Let $\{a_k\}$ be a sequence of real numbers. For each natural number n, let**

$$s_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Suppose that the sequence $\{a_k\}$ converges to the real number a. Prove that the sequence $\{s_n\}$ also converges to a. Give a complete ε style proof. Write in complete sentences.

Fix $\varepsilon > 0$. The sequence $\{a_k\}$ converges to a so there exists a natural number k_0 so that if $k > k_0$, then $|a_k - a| < \frac{\varepsilon}{2}$. Let B equal the fixed number $B = |(\sum\limits_{k=1}^{k_0} a_k) - k_0 a|$. Pick a natural number n_1 with $\frac{2B}{\varepsilon} < n_1$. Let $n_0 = \max\{n_1, k_0\}$. If $n_0 < n$, then

$$|s_n - a| = \left|\frac{a_1 + a_2 + \cdots + a_n}{n} - a\right| = \left|\frac{a_1 + a_2 + \cdots + a_n - na}{n}\right| =$$

$$= \left|\frac{(a_1 + a_2 + \cdots + a_{k_0} - k_0 a) + (a_{k_0+1} - a) + (a_{k_0+2} - a) + \cdots + (a_n - a)}{n}\right|.$$

Use the triangle inequality to see that

$$|s_n - a| \leq \frac{|a_1 + a_2 + \cdots + a_{k_0} - k_0 a|}{n} + \frac{|a_{k_0+1} - a|}{n} + \frac{|a_{k_0+2} - a|}{n} + \cdots + \frac{|a_n - a|}{n}.$$

The first term on the right of the sign $\leq$ is $\frac{B}{n}$. Our choice of n_0 ensures that $\frac{B}{n} < \frac{\varepsilon}{2}$. If ℓ is a positive integer, then our choice of n_0 ensures that $|a_{k_0+\ell} - a| \leq \frac{\varepsilon}{2}$. At this point we have

$$|s_n - a| \leq \frac{\varepsilon}{2} + (n - k_0)\frac{\frac{\varepsilon}{2}}{n} \leq \frac{\varepsilon}{2} + n\frac{\frac{\varepsilon}{2}}{n} = \varepsilon.$$

We have shown that $n > n_0 \implies |s_n - a| < \varepsilon$. We conclude that the sequence $\{s_n\}$ converges to a.