Math 554, Exam 3, Summer 2005, solution

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

There are 6 problems. Problems 1 through 2 are worth 9 points each. Problems 3 through 6 are worth 8 points each. The exam is worth a total of 50 points.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after the class is finished.

1. **For each natural number n, let**

$$s_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.$$

Prove that $\{s_n\}$ is a Cauchy sequence.

Let $n < m$. We see that

$$|s_m - s_n| = \left| \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \right|.$$

Observe that $\frac{1}{\ell!} < \frac{1}{2^{\ell-1}}$ because $2^{\ell-1} \le 1 \cdot 2 \cdot 3 \cdot \ldots \cdot \ell$. It follows that

$$|s_m - s_n| \le \left| \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} \right|.$$

If r is any real number other than 1, and $S = r^n + r^{n+1} + \cdots + r^{m-1}$, then

$$S - rS = r^n + r^{n+1} + \cdots + r^{m-1} - (r^{n+1} + \cdots + r^m) = r^n - r^m;$$

hence,

$$S = \frac{r^n - r^m}{1 - r}.$$

We see that

$$|s_m - s_n| \le \frac{\frac{1}{2^n} - \frac{1}{2^m}}{1 - \frac{1}{2}} = 2\left(\frac{1}{2^n} - \frac{1}{2^m}\right) \le 2\frac{1}{2^n} = \frac{1}{2^{n-1}}.$$

(The inequality on the right used the fact that $0 < \frac{1}{2^m} < \frac{1}{2^n}$.)

Given $\varepsilon > 0$, pick n_0 so large that $\frac{1}{2^{n_0-1}} < \varepsilon$. If n and m are any integers greater than n_0, then $\min\{n, m\} > n_0$, and

$$|s_n - s_m| \le \frac{1}{2^{\min\{m,n\}-1}} < \frac{1}{2^{n_0-1}} < \varepsilon.$$

The proof is complete.

2. **Let A be a set. For each $a \in A$, let U_a be an open subset of $\mathbb{R}$.**
   (a) Does $\bigcup_{a\in A} U_a$ have to be an open set? If yes, prove the statement? If no, give a counterexample.
   (b) Does $\bigcap_{a\in A} U_a$ have to be an open set? If yes, prove the statement? If no, give a counterexample.

The answer to (a) is YES. If $p \in \bigcup_{a\in A} U_a$, then $p \in U_{a_0}$ for some $a_0 \in A$; hence, there exists $\varepsilon > 0$ with $N_\varepsilon(p) \subseteq U_{a_0}$. It follows that $N_\varepsilon(p) \subseteq \bigcup_{a\in A} U_a$ and $\bigcup_{a\in A} U_a$ is an open set.

The answer to (b) is NO. Let $A = \mathbb{N}$. For each $n \in \mathbb{N}$, let $U_n = (-\frac{1}{n}, \frac{1}{n})$. We see that each U_n is an open set; but that $\bigcap_{n=1}^{\infty} U_n = \{0\}$; which is not an open set.

3. **Define** *the sequence converges*. **Use complete sentences. Include everything that is necessary, but nothing more.**

The sequence of real numbers $\{a_n\}$ converges to the real number p if for all $\varepsilon > 0$, there exists an integer n_0 such that whenever n is an integer with $n > n_0$, then $|a_n - p| < \varepsilon$.

4. **For each natural number $n \in \mathbb{N}$, let K_n be a closed set of the form $(-\infty, b_n)$ for some $b_n \in \mathbb{R}$. Assume $K_n \supseteq K_{n+1}$ for all n. Does $\bigcup_{n=1}^{\infty} K_n$ have to be non-empty? If yes, prove the statement? If no, give a counterexample.**

OOPS!. This problem is riddled with typos. The set $K_n = (-\infty, b_n)$ is not a closed set. However, in any event, if $K_n \supseteq K_{n+1}$ for all n, then $\bigcup_{n=1}^{\infty} K_n = K_1$ and the set $K_1 = (-\infty, b_1)$ is not empty; indeed, $b_1 - 1 \in K_1$; so the union $\bigcup_{n=1}^{\infty} K_n$ is also not empty because $b_1 - 1 \in \bigcup_{n=1}^{\infty} K_n$. One should probably state that $\bigcup_{n=1}^{\infty} K_n$ has to be non-empty; justify this statement; and then move on.

5. **Consider the sequence $\{a_n\}$ with $a_1 = 10$ and, for all $n \geq 2$, $a_n = \frac{1}{2}(a_{n-1} + \frac{7}{a_{n-1}})$.**
   (a) Show that this sequence is bounded below by $\sqrt{7}$.
   (b) Show that the sequence is a decreasing sequence.

We prove (a) by induction. We see that $\sqrt{7} < a_1$. For $n \geq 2$, we assume that $\sqrt{7} < a_{n-1}$. We prove $\sqrt{7} < a_n$. Observe that

$$0 \leq (a_{n-1} - \sqrt{7})^2 = a_{n-1}^2 - 2\sqrt{7}a_{n-1} + 7.$$

It follows that

$$2\sqrt{7}a_{n-1} \leq a_{n-1}^2 + 7.$$

Divide both sides by $2a_{n-1}$, which we know is positive (by induction), to see

$$\sqrt{7} < \frac{1}{2}\left(a_{n-1} + \frac{7}{a_{n-1}}\right) = a_n.$$

Now we do (b). We saw in (a) that $\sqrt{7} < a_{n-1}$, whenever $2 \leq n$. The numbers $\sqrt{7}$ and a_{n-1} are positive; hence, it follows that $7 \leq a_{n-1}^2$. Divide both sides by the positive number a_{n-1} to see that

$$\frac{7}{a_{n-1}} \leq a_{n-1}.$$

Add a_{n-1} to both sides to see

$$a_{n-1} + \frac{7}{a_{n-1}} \leq 2a_{n-1}.$$

Divide by 2 to get

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{7}{a_{n-1}}\right) \leq a_{n-1}.$$

6. **Consider the sequence $\{a_n\}$ with $a_1 = \frac{1}{4}$ and, for all $n \geq 2$, $a_n = \frac{1}{3}(1 - a_{n-1}^3)$.**
   (a) Show that $0 < a_n < \frac{1}{3}$, for all n.
   (b) Prove that $\{a_n\}$ is a contractive sequence.

We prove (a) by induction. We see that $0 < a_1 < \frac{1}{3}$. Our induction hypothesis is that $0 < a_{n-1} < \frac{1}{3}$. We prove $0 < a_n < \frac{1}{3}$. The induction hypothesis ensures that $0 < a_{n-1}^3 < \frac{1}{27}$; hence, $1 - \frac{1}{27} < 1 - a_{n-1}^3 < 1$; so, $0 < 1 - a_{n-1}^3 < 1$. It follows that $0 < \frac{1}{3}(1 - a_{n-1}^3) < \frac{1}{3}$. We have established that $0 < a_n < \frac{1}{3}$.

For (b), we compare $|a_{n+1} - a_n|$ and $|a_n - a_{n-1}|$. We see that

$$|a_{n+1} - a_n| = \left|\tfrac{1}{3}(1 - a_n^3) - \tfrac{1}{3}(1 - a_{n-1}^3)\right| = \tfrac{1}{3}\left|(1 - a_n^3) - (1 - a_{n-1}^3)\right| = \tfrac{1}{3}|a_{n-1}^3 - a_n^3|.$$

We know how to factor the difference of perfect cubes. (Do a long division, if necessary.)

$$\begin{aligned}|a_{n+1} - a_n| &= \tfrac{1}{3}\left|(a_{n-1} - a_n)(a_{n-1}^2 + a_{n-1}a_n + a_n^2)\right| \\ &= \tfrac{1}{3}|a_{n-1} - a_n||a_{n-1}^2 + a_{n-1}a_n + a_n^2|.\end{aligned}$$

Use the triangle inequality to see

$$|a_{n+1} - a_n| \le \tfrac{1}{3}|a_{n-1} - a_n|(|a_{n-1}|^2 + |a_{n-1}||a_n| + |a_n|^2|).$$

Use part (a) to see

$$|a_{n+1} - a_n| \le \tfrac{1}{3}|a_{n-1} - a_n|(|\tfrac{1}{3^2} + \tfrac{1}{3}\tfrac{1}{3} + \tfrac{1}{3^2}|) = \tfrac{1}{3}|a_{n-1} - a_n|(\tfrac{3}{9}).$$

We have shown that

$$|a_{n+1} - a_n| \le \tfrac{1}{9}|a_{n-1} - a_n|.$$

We have shown that $\{a_n\}$ is a contractive sequence.