Math 554, Exam 2, Summer 2005 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

There are 9 problems. Problems 1 through 4 are worth 5 points each. Problems 5 through 9 are worth 6 points each. The exam is worth a total of 50 points.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. **If you are interested, be sure to tell me.**

I will post the solutions on my website shortly after the class is finished.

1. **Define** *upper bound*. **Use complete sentences. Include everything that is necessary, but nothing more.**

The real number u is an *upper bound* of the non-empty set of real numbers E if $e \leq u$ for all $e \in E$.

2. **Define** *supremum*. **Use complete sentences. Include everything that is necessary, but nothing more.**

The real number α is the *supremum* of the non-empty set of real numbers E if α is an upper bound of E and if d is a real number with $d < \alpha$, then d is not an upper bound of E.

3. **Define** *the sequence converges*. **Use complete sentences. Include everything that is necessary, but nothing more.**

The sequence of real numbers $\{a_n\}$ converges to the real number p if for all $\varepsilon > 0$, there exists an integer n_0 such that whenever n is an integer with $n > n_0$, then $|a_n - p| < \varepsilon$.

4. **State the Nested Interval Property. Use complete sentences. Include everything that is necessary, but nothing more.**

If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a countable family of closed bounded intervals, then the intersection $\bigcap_{n=1}^{\infty} I_n$ is not empty.

5. **Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Suppose that $\{a_n\}$ converges to the real number a and $\{b_n\}$ converges to the real number b. Prove that the sequence $\{a_nb_n\}$ converges to ab. ("We did this in class" is not a satisfactory answer. I expect a complete, coherent proof.)**

Let $\varepsilon > 0$ be arbitrary, but fixed.

• The sequence $\{a_n\}$ converges to a, so there exists n_1 such that if $n \geq n_1$, then $|a_n - a| \leq 1$. For such n, the Corollary to the triangle inequality tells us that

$$|a_n| - |a| \leq ||a_n| - |a|| \leq |a_n - a| \leq 1;$$

and therefore, $|a_n| \leq |a| + 1$.

• The sequence $\{b_n\}$ converges to b, so there exists n_2 such that if $n \geq n_2$, then $|b_n - b| \leq \frac{\varepsilon}{2(|a|+1)}$.

• The sequence $\{a_n\}$ converges to a, so there exists n_3 such that if $n \geq n_3$, then $|a_n - a| \leq \frac{\varepsilon}{2(|b|+1)}$.

Let n_0 be the maximum of the three integers n_1, n_2, and n_3. Take $n \geq n_0$. We know that $n \geq n_1$; and therefore,

$$|a_n| \leq |a| + 1. \tag{1}$$

We know that $n \geq n_2$; and therefore,

$$|b_n - b| \leq \frac{\varepsilon}{2(|a| + 1)}. \tag{2}$$

We know that $n \geq n_3$; and therefore,

$$|a_n - a| \leq \frac{\varepsilon}{2(|b| + 1)}. \tag{3}$$

The triangle inequality tells us that

$$|a_nb_n - ab| = |(a_nb_n - a_nb) + (a_nb - ab)| \leq |a_nb_n - a_nb| + |a_nb - ab| = |a_n||b_n - b| + |a_n - a||b|.$$

Use (1), (2) and (3) to see that

$$|a_nb_n - ab| \leq |a_n||b_n - b| + |a_n - a||b| \leq (|a| + 1)\frac{\varepsilon}{2(|a| + 1)} + \frac{\varepsilon}{2(|b| + 1)}|b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

6. **Give an example of a set X and a function $f\colon X \to X$ with f one-to-one, but f not onto.**

Let X be the set of natural numbers $\{1, 2, 3, 4, \ldots\}$, and let $f\colon X \to X$ be $f(n) = n + 1$. It is clear that f is one-to-one, but f is not onto because there is no element n in X with $f(n) = 1$.

7. **Find $\bigcap_{n=1}^{\infty} [-n, n]$.**

The intersection is $[-1, 1]$. The intervals in question are

$$[-1, 1] \subset [-2, 2] \subset [-3, 3] \subset \ldots .$$

The interval $[-1, 1]$ is contained in all of the other intervals and nothing else is.

8. **Suppose that A and B are non-empty sets of real numbers. Suppose further that 1 is a lower bound for both A and B. Let**

$$C = \{ab \mid a \in A \textbf{ and } b \in B\}.$$

Prove $\inf C = (\inf A)(\inf B)$**.**

Let $\alpha = \inf A$, $\beta = \inf B$, and $\gamma = \inf C$.

We first show that $\alpha\beta \le \gamma$. We start be showing that $\alpha\beta$ is a lower bound for C. Take an arbitrary element c from C. This c is equal to ab for some $a \in A$ and $b \in B$. We know that α is a lower bound for A and β is a lower bound for B; so, $\alpha \le a$ and $\beta \le b$. Every number in sight is positive; therefore, $\alpha\beta \le ab = c$. We have shown that $\alpha\beta$ is a lower bound for C. The number γ is the greatest lower bound for C. We conclude that $\alpha\beta \le \gamma$.

Now we show that $\alpha\beta < \gamma$ is not possible. This part of our argument is a proof by contradiction. We suppose that $\alpha\beta < \gamma$. We show that this supposition leads to a contradiction. At any rate, we suppose $\alpha\beta < \gamma$. It follows that $\alpha < \gamma/\beta$ (again all numbers in sight are positive). The number γ/β is too big to be a lower bound for A; hence there is an element $a \in A$ with $a < \gamma/\beta$. It follows that $\beta < \gamma/a$. The number γ/a is too big to be a lower bound for B; hence there is an element $b \in A$ with $b < \gamma/a$. It follows that ab, which is an element of C, is SMALLER than γ which is $\inf C$. This is not possible. This contradiction results from the supposition that $\alpha\beta < \gamma$. Hence, $\alpha\beta < \gamma$ is not possible.

We have shown that $\alpha\beta \le \gamma$ does happen. We have also shown that $\alpha\beta < \gamma$ does not happen. The only remaining possibility is that $\alpha\beta = \gamma$.

9. **Let S be a set of real numbers. Let p be a limit point of S. Prove that there exists a sequence a_n IN S which converges to p.**

Let a_1 be any point in S, other than p. Let $\varepsilon_1 = |a_1 - p|/2$. The point p is a limit point of S; so the ε_1-neighborhood of p must contain a point of S other than p; call this point a_2. Notice that

$$|a_2 - p| \le \frac{|a_1 - p|}{2}.$$

Let $\varepsilon_2 = |a_2 - p|/2$. We repeat the above thought process. The point p is a limit point of S; so, the ε_2-neighborhood of p must contain a point of S other than p; call this point a_3. Notice that

$$|a_3 - p| \le \frac{|a_2 - p|}{2} \le \frac{|a_1 - p|}{4}.$$

Continue in this manner to manufacture $a_1, a_2, a_3, \ldots$ in S. None of theses numbers is equal to p and

$$|a_n - p| \le \frac{|a_1 - p|}{2^{n-1}}.$$

It is now clear that the sequence $\{a_n\}$ converges to p. Indeed, if $\varepsilon > 0$ is given, then take n_0 with $\frac{|a_1-p|}{2^{n_0-1}} < \varepsilon$. We see that if $n > n_0$, then

$$|a_n - p| \le \frac{|a_1 - p|}{2^{n-1}} < \frac{|a_1 - p|}{2^{n_0-1}} < \varepsilon.$$