Math 554, Exam 3 Solutions, Summer 2004

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ... ; although, by using enough paper, you can do the problems in any order that suits you.

There are 9 problems. Problems 1 through 5 are worth 6 points each. Problems 6 through 9 are worth 5 points each. The exam is worth a total of 50 points.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Define** *Cauchy sequence*. **Use complete sentences.**

The sequence $\{a_n\}$ is a a *Cauchy sequence* if for all $\varepsilon > 0$, there exists n_0 such that whenever $n, m > n_0$, then $|a_n - a_m| < \varepsilon$.

2. **Define** *limit point*. **Use complete sentences.**

The real number p is a *limit point* of the set of real numbers E if, for all $\varepsilon > 0$, there exists $q \in E$ with $q \neq p$ and $|q - p| \leq \varepsilon$.

3. **For each natural number n, let U_n be an open set in $\mathbb{R}$. Is the intersection $\bigcap_{n=1}^{\infty} U_n$ always an open set? If yes, prove the result. If no, give a counterexample.**

NO! Let U_n be the open interval $(-\frac{1}{n}, \frac{1}{n})$, for each natural number n. We see that each U_n is open, but the intersection $\bigcap_{n=1}^{\infty} U_n$ is equal to $\{0\}$, which is not open.

4. **For each natural number n, let U_n be an open set in $\mathbb{R}$. Is the union $\bigcup_{n=1}^{\infty} U_n$ always an open set? If yes, prove the result. If no, give a counterexample.**

YES! If $p \in \bigcup_{n=1}^{\infty} U_n$, then $p \in U_{n_1}$ for some fixed n_1; hence, there exists ε such that $N_\varepsilon(p) \subseteq U_{n_1}$. It follows that $N_\varepsilon(p) \subseteq \bigcup_{n=1}^{\infty} U_n$; and therefore, $\bigcup_{n=1}^{\infty} U_n$ is an open subset of $\mathbb{R}$.

5. **State the theorem which characterizes the closed sets of $\mathbb{R}$ in terms of information about the limit points.**

The subset K of $\mathbb{R}$ is closed if and only if K contains all of its limit points.

6. **Prove the first version of the Bolzano-Weierstrass Theorem. That is, prove that every bounded infinite subset of $\mathbb{R}$ has a limit point.**

Let S be a bounded infinite subset of $\mathbb{R}$, and let I be a finite closed interval which contains S. Cut I in half. At least one of the resulting two closed subintervals of I contains infinitely many elements of S. Call this interval I_1. Continue in this manner to build the closed interval I_n, for each natural number n, with the length of I_n equal to $1/2^n$ times the length of I and I_n contains infinitely many elements of S. The nested interval property of $\mathbb{R}$ tells us that the intersection $\bigcap_{n=1}^{\infty} I_n$ is non-empty. Let p be an element of $\bigcap_{n=1}^{\infty} I_n$. We will show that p is a limit point of S. Given $\varepsilon > 0$, there exists n large enough that the length of I_n is less than ε. We know that $p \in I_n$. It follows that $I_n \subseteq N_\varepsilon(p)$. Furthermore, there is at least one element q of S with $q \neq p$ and $q \in N_\varepsilon(p)$; since $I_n \cap S$ is infinite.

7. **Let $\{p_n\}$ be a bounded sequence of real numbers and let $p \in \mathbb{R}$ be such that every convergent subsequence of $\{p_n\}$ converges to p. Prove that the sequence $\{p_n\}$ converges to p.**

Suppose that

$$\text{the sequence } \{p_n\} \text{ does NOT converge to } p. \tag{6}$$

In this case, there exists $\varepsilon > 0$ such that

$$\text{for every } n_0 \in \mathbb{N} \text{ there exists } n > n_0 \text{ such that } |p_n - p| \geq \varepsilon. \tag{7}$$

Apply (7) to find $n_1 > 1$, with $|p_{n_1} - p| > \varepsilon$. Apply (7) to find $n_2 > n_1$, with $|p_{n_2} - p| > \varepsilon$. Apply (7) to find $n_3 > n_2$, with $|p_{n_3} - p| > \varepsilon$. Continue in this manner to construct a subsequence

$$p_{n_1}, p_{n_2}, p_{n_3}, \dots \tag{8}$$

of the original sequence $\{p_n\}$ which never gets closer to p than ε. The Bolzano-Weierstrass Theorem (version 2) guarantees that some subsequence of (8) converges. This subsequence of (8) does not converge to p because the subsequence never gets within ε of p. On the other hand, subsequence of (8) is also a subsequence of the original sequence $\{p_n\}$; and therefore, must converge to p by the original hypothesis. This is a contradiction. The original supposition (6) must be false. We conclude that the sequence $\{p_n\}$ does converge to p.

8. **Let a_1 be a real number in the open interval $(0,1)$. Define the sequence $\{a_n\}$ by $a_{n+1} = \frac{1}{5}(1 - a_n^3)$, for all $n \geq 1$. Prove that the sequence $\{a_n\}$ is a contractive sequence.**

Observe that

$$\left|\frac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n}\right| = \left|\frac{\frac{1}{5}(1 - \left(\frac{1}{5}(1 - a_n^3)\right)^3) - \frac{1}{5}(1 - a_n^3)}{\frac{1}{5}(1 - a_n^3) - a_n}\right|.$$

Multiply top and bottom by 5 to get

$$= \left| \frac{(1 - \left(\frac{1}{5}(1 - a_n^3)\right)^3) - (1 - a_n^3)}{(1 - a_n^3) - 5a_n} \right| = \left| \frac{1 - \left(\frac{1}{5}(1 - a_n^3)\right)^3 - 1 + a_n^3}{(1 - a_n^3) - 5a_n} \right|$$

$$= \left| \frac{- \left(\frac{1}{5}(1 - a_n^3)\right)^3 + a_n^3}{(1 - a_n^3) - 5a_n} \right|.$$

Pull $\frac{-1}{125}$ out of the numerator to get

$$= \left| \frac{-1}{125} \right| \left| \frac{(1 - a_n^3)^3 - (5a_n)^3}{(1 - a_n^3) - 5a_n} \right|.$$

The numerator is the difference of perfect cubes. (If you don't remember the formula for the difference of perfect cubes, then just divide $A^3 - B^3$ by $A - B$ to find the other factor; use long division. At any rate, $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$.) At this point, we have

$$\left| \frac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n} \right| = \frac{1}{125} \left| \frac{((1 - a_n^3) - 5a_n)((1 - a_n^3)^2 + (1 - a_n^3)5a_n + (5a_n)^2)}{(1 - a_n^3) - 5a_n} \right|.$$

The factor on the left of the numerator is exactly equal to the denominator; so

$$\left| \frac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n} \right| = \frac{1}{125} \left| (1 - a_n^3)^2 + (1 - a_n^3)5a_n + (5a_n)^2 \right|$$

$$= \frac{1}{125} \left| 1 - 2a_n^3 + a_n^6 + 5a_n - 5a_n^4 + 25a_n^2 \right|.$$

Use the triangle inequality to see that

$$\left| \frac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n} \right| \le \frac{1}{125}(|1| + 2|a_n^3| + |a_n^6| + 5|a_n| + 5|a_n^4| + 25|a_n^2|).$$

Induction shows that each number a_n is in the open interval $(0, 1)$. Indeed, $a_1 \in (0, 1)$, and if $a_{n-1} \in (0, 1)$, then $a_{n-1}^3 \in (0, 1)$; so, $1 - a_{n-1}^3 \in (0, 1)$, and $a_n = \frac{1}{5}(1 - a_{n-1}^3) \in (0, 1)$. Thus,

$$\left| \frac{a_{n+2} - a_{n+1}}{a_{n+1} - a_n} \right| \le \frac{1}{125}(1 + 2 + 1 + 5 + 5 + 25) = \frac{39}{125}.$$

We have shown that

$$|a_{n+2} - a_{n+1}| \le \frac{39}{125}|a_{n+1} - a_n|,$$

for all n . The number $b = \frac{39}{125}$ is between 0 and 1 . Thus, $\{a_n\}$ is a contractive sequence.

9. **Let K be a closed non-empty subset of $\mathbb{R}$ and let x be an element of $\mathbb{R}$, with $x \notin K$. Prove that there exists at least one element y of K which is closest to x. In other words, if $z \in K$, then $|x-y| \leq |x-z|$.**

Let $D = \{|x-z| \mid z \in K\}$. The set D is bounded below by 0; so this set has an infimum d in $\mathbb{R}$. Our job is to show that there is an element y of K with $|x-y| = d$. The fact that d is the infimum of D ensures that for each natural number n, there is $z_n \in K$, with $|x-z_n| < d + \frac{1}{n}$. The sequence $\{z_n\}$ is bounded, since each z_n is always within 1 of x; so the Bolzano-Weierstrass Theorem (version 2) ensures that some subsequence of $\{z_n\}$ converges. Suppose that the subsequence converges to y. It is clear that $|x-y| = d$. Either the tail end of the subsequence is constant (in which case $y = z_n \in K$ for infinitely many n), or y is a limit point of K. The set K is closed; so K contains all of its limit points. In any event, $y \in K$ and the proof is complete.