Math 5§4 Summer 2000 Exam 4

① The subset E of $\mathbb{R}$ is an <u>open set</u> if for all $x \in E$ there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq E$.

② The subset E of $\mathbb{R}$ is a <u>compact set</u> if every open cover of E contains a finite sub cover of E.

③ Let E be a subset of $\mathbb{R}$, f be a function from E to $\mathbb{R}$, p be a limit point of E and L be an element of $\mathbb{R}$. Then $\lim_{x \to p} f(x) = L$ means for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in E$ with $0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

④ The set K is not a closed set, so K has a limit x which is not in K. x is a limit point of K; so there is a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in K$ for all k and the limit of the sequence is x. Consider the set $E = \{x_n \mid n \in \mathbb{N}\}$. The only limit point of E is x.
For each $y \in K$, $y \neq x$ so y is not a limit point of E
So $\exists \varepsilon_y > 0$ with $N_{\varepsilon_y}(y) \cap E$ contains at most one element (if $y \in E$, then $N_{\varepsilon_y}(y) \cap E$ has exactly one element
if $y \notin E$, then $N_{\varepsilon_y}(y) \cap E$ has no elements.)

$\mathcal{U} = \{N_{\varepsilon_y}(y) \mid y \in K\}$ covers K

But No finite subset of $\mathcal{U}$ covers the infinite set E. So surely no finite subset of $\mathcal{U}$ covers K (because $E \subseteq K$). Thus K is not compact.

⑤ Given $\varepsilon > 0$. Observe
$$|f(x) - L| = |(4x-7) - 5| = |4x-12| = |4(x-3)| = 4|x-3|$$
where $f(x) = 4x - 7$ and $L = 5$. If $\delta = \frac{\varepsilon}{4}$, then $x \in \mathbb{R}$ $0 < |x-3|$
$$\Rightarrow |f(x) - L| = 4|x-3| < 4\delta = 4\tfrac{\varepsilon}{4} = \varepsilon.$$