Math 544 Final Exam Summer 2000

Use the paper provided. Problem 10 is worth 12 points. Problem 14 is worth 10 points. The other problems are worth 6 points each.

If you e-mail me requesting that I send you your grade, I will do it. Or, get your grade from VIP or TIPS.

1. Define “open set”.

2. Define “compact set”.

3. Let \( f: E \to \mathbb{R} \) be a function which is defined on a subset \( E \) of \( \mathbb{R} \). Define \( \lim_{x \to p} f(x) = L \). (Be sure to tell me what kind of a thing \( p \) is, and what kind of a thing \( L \) is.)

4. Let \( a_1, a_2, \ldots \) be a sequence of real numbers, and let \( L \) be a real number. Define the expression, “the limit of the sequence \( \{a_n\}_{n=1}^\infty \) is equal to \( L \”).

5. Define “supremum”.

6. Define “limit point”.

7. State the Heine-Borel Theorem.

8. State either version of the Bolzano-Weierstrass Theorem.

9. State the least upper bound property of the real numbers.

10. Let \( f(x) = \begin{cases} 2x^2 & \text{if } x \text{ is rational} \\ 5x - 2 & \text{if } x \text{ is irrational.} \end{cases} \)
    
    (a) Does \( \lim_{x \to 2} f(x) \) exist? If the limit exists, find the limit and prove your answer. If the limit does not exist, give a coherent proof why it doesn’t exist.
    
    (b) Does \( \lim_{x \to 1} f(x) \) exist? If the limit exists, find the limit and prove your answer. If the limit does not exist, give a coherent proof why it doesn’t exist.

11. Prove that the continuous image of a compact set is compact.

12. Let \( A \) be a set. For each \( a \in A \), let \( E_a \) be an open subset of \( \mathbb{R} \). Is the union \( \bigcup_{a \in A} E_a \) always an open set? If your answer is yes, prove it. If your answer is no, give an example.

13. Let \( A \) be a set. For each \( a \in A \), let \( E_a \) be an open subset of \( \mathbb{R} \). Is the intersection \( \bigcap_{a \in A} E_a \) always an open set? If your answer is yes, prove it. If your answer is no, give an example.
14. Let $A$ and $B$ be non-empty sets, and let $f: A \rightarrow B$ and $g: B \rightarrow A$ be functions. Suppose that $g \circ f$ is the identity function on $A$. (In other words, $g(f(a)) = a$ for all $a$ in $A$.)

(a) Prove that $f$ is one-to-one.

(b) Prove that $g$ is onto.

(c) Give an example to show that $g$ does not have to be one-to-one. (Your example can be very small.)

15. Let $f$ and $g$ be functions from $\mathbb{R}$ to $\mathbb{R}$, and let $p$, $A$, and $B$ be real numbers. Suppose that $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$. Prove $\lim_{x \to p} f(x)g(x) = AB$. 