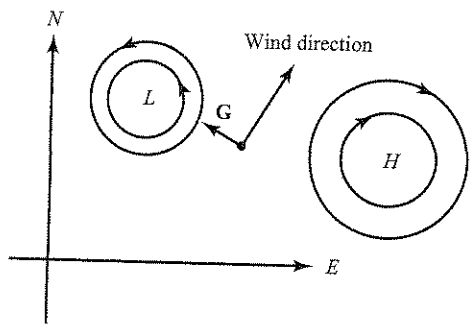
29. $\nabla f = (2xe^{x^2} + y^2 \sin xy^2, 2xy \sin xy^2, 0)$; check that $\nabla \times \nabla f = 0$ from this.

31. (a) $(yz^2, xz^2, 2xyz)$;
(b) $(z - y, 0, -x)$
(c) $(2xyz^3 - 3xy^2z^2, 2x^2y^2z - y^2z^3, y^2z^3 - 2x^2yz^2)$

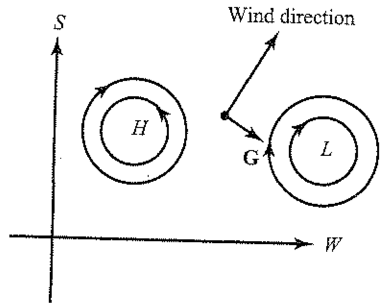
33. $\operatorname{div} \mathbf{F} = 0$; $\operatorname{curl} \mathbf{F} = (0, 0, 2(x^2+y^2)f'(x^2+y^2) + 2f(x^2+y^2))$

35. (a) A cone about $\mathbf{i}'$ making an angle of $\pi/3$ with $\mathbf{i}'$.
(b) $\nabla g = (3x^2, 5z, 5y + 2z)$

37. (a) $[\partial P/\partial x)^2 + (\partial P/\partial y)^2]^{1/2}$
(b) A small packet of air would obey $\mathbf{F} = m\mathbf{a}$.
(c)



(d)



39. (a) $\dfrac{\sqrt{R^2+\rho^2}}{\rho}(z_0 - z_1)$
(b) $\sqrt{\dfrac{2(R^2+\rho^2)z_0}{g\rho^2}}$

41. 680 miles per hour

Chapter 5

Section 5.1

1. (a) 1
(b) 2
(c) $\ln 128 + \ln \sqrt{2}$
(d) $\dfrac{1}{2} \ln 2 = \ln \sqrt{2}$

3. (a) $\dfrac{13}{15}$ (b) $\pi + \dfrac{1}{2}$
(c) 1 (d) $\log 2 - \dfrac{1}{2}$

5. To show that the volumes of the two cylinders are equal, show that their area functions are equal.

7. $2r^3(\tan\theta)/3$

9. $\dfrac{26}{9}$

11. $(2/\pi)(e^2 + 1)$

13. $\dfrac{35795}{8}$

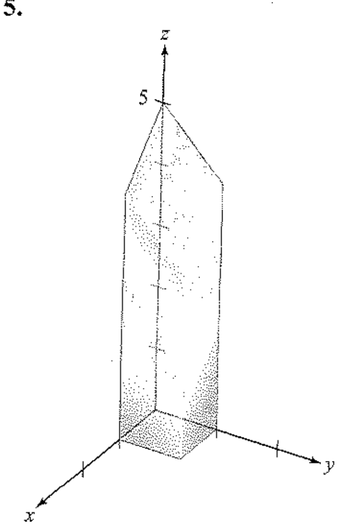
15. $\dfrac{196}{15}$

Section 5.2

1. (a) $\dfrac{7}{12}$ (b) $e - 2$
(c) $\dfrac{1}{9}\sin 1$ (d) $2\ln 4 - 2$

3. 0

5.



7. 1/4

9. Use Fubini's theorem to write

$$\iint_R [f(x)g(y)]\,dx\,dy = \int_c^d g(y)\left[\int_a^b f(x)\,dx\right]dy,$$

and notice that $\int_a^b f(x)\,dx$ is a constant and so may be pulled out.

11. 11/6

13. By Exercise 2(a), we have:

$$f(m,n) = \iint_R x^m y^n\,dx\,dy = \left(\frac{1}{m+1}\right)\left(\frac{1}{n+1}\right).$$

Then, as $m, n \to \infty$, we see that $\lim f(m,n) = 0$.

15. Because $\int_0^1 dy = \int_0^1 2y\,dy = 1$, we have $\int_0^1 [\int_0^1 f(x,y)\,dy]\,dx = 1$. In any partition of $R = [0,1] \times [0,1]$, each rectangle R_{jk} contains points $\mathbf{c}_{jk}^{(1)}$ with x rational and $\mathbf{c}_{jk}^{(2)}$ with x irrational. If in the regular partition of order n, we choose $\mathbf{c}_{jk} = \mathbf{c}_{jk}^{(1)}$ in those rectangles with $0 \le y \le \frac{1}{2}$ and $\mathbf{c}_{jk} = \mathbf{c}_{jk}^{(2)}$ when $y > \frac{1}{2}$, the approximating sums are the same as those for

$$g(x,y) = \begin{cases} 1 & 0 \le y \le \frac{1}{2} \\ 2y & \frac{1}{2} < y < 1. \end{cases}$$

Because g is integrable, the approximating sums must converge to $\int_R g\,dA = 7/8$. However, if we had picked all $\mathbf{c}_{ij} = \mathbf{c}_{jk}^{(1)}$, all approximating sums would have the value 1.

17. Fubini's theorem does not apply because the integrand is not continuous nor bounded at (0, 0).

Section 5.3

1. (a) (iii) (b) (iv)
(c) (ii) (d) (i)

3. (a) 1/3, both.
(b) 5/2, both.
(c) $(e^2 - 1)/4$, both.
(d) 1/35, both.

5. $A = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\,dx = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = r^2[\arcsin 1 - \arcsin(-1)] = \pi r^2$

7. 28,000 ft^3

9. 0

11. y-simple; $\pi/2$.

13. $\dfrac{2}{3}$

15. 50π

17. $\pi/24$

19. Compute the integral with respect to y first. Split that into integrals over $[-\phi(x), 0]$ and $[0, \phi(x)]$ and change variables in the first integral, or use symmetry.

21. Let $\{R_{ij}\}$ be a partition of a rectangle R containing D and let f be 1 on D. Thus, f^* is 1 on D and 0 on $R \backslash D$. Let $\mathbf{c}_{jk} \in R\backslash D$ if R_{ij} is not wholly contained in D. The approximating Riemann sum is the sum of the areas of those rectangles of the partition that are contained in D.

Section 5.4

1. (a) $\int_0^4 \int_0^{2x} dy\,dx$
(b) $\int_0^3 \int_{y^2}^{9} dx\,dy$
(c) $\int_{-4}^{4} \int_0^{\sqrt{16-x^2}} dy\,dx$
(d) $\int_0^1 \int_{\frac{\pi}{2}}^{\arcsin y} dx\,dy$

3. (a) 1/8 (b) $\pi/4$ (c) 17/12
(d) $G(b) - G(a)$, where $dG/dy = F(y,y) - F(a,y)$ and $\partial F/\partial x = f(x,y)$.

5. $\dfrac{1}{3}(e-1)$

7. Note that the maximum value of f on D is e and the minimum value of f on D is $1/e$. Use the ideas in the proof of Theorem 4 to show that

$$\frac{1}{e} \le \frac{1}{4\pi^2} \iint f(x,y)\,dA \le e.$$

9. The smallest value of $f(x,y) = 1/(x^2 + y^2 + 1)$ on D is $\dfrac{1}{6}$, at (1, 2), and so

$$\iint_D f(x,y)\,dx\,dy \ge \frac{1}{6} \cdot \text{area } D = 1.$$

The largest value is 1, at (0, 0), and so

$$\iint_D f(x,y)\,dx\,dy \le 1 \cdot \text{area } D = 6.$$

11. $\dfrac{4}{3}\pi abc$

13. $\pi(20\sqrt{10}-52)/3$

15. $\sqrt{3}/4$

17. D looks like a slice of pie.

$$\int_0^1\left[\int_0^x f(x,y)\,dy\right]dx+\int_1^{\sqrt{2}}\left[\int_0^{\sqrt{2-x^2}} f(x,y)\,dy\right]dx$$

19. Use the chain rule and the fundamental theorem of calculus.

Section 5.5

1. (a) (ii) (b) (i) (c) (iii) (d) (iv)

3. 1/3

5. 10

7. $x^2+y^2\le z\le\sqrt{x^2+y^2}$, $-\sqrt{1-y^2}\le x\le\sqrt{1-y^2}$, $-1\le y\le 1$

9. $0\le z\le\sqrt{1-x^2-y^2}$, $-\sqrt{1-y^2}\le x\le\sqrt{1-y^2}$, $-1\le y\le 1$

11. $50\pi/\sqrt{6}$

13. 1/2

15. 0

17. $a^5/20$

19. 0

21. 3/10

23. 1/6

25. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dz\,dy\,dx$

27. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$

29. $\iint_D\int_0^{f(x,y)} dz\,dx\,dy=\iint_D f(x,y)\,dx\,dy$

31. Let M_ϵ and m_ϵ be the maximum and minimum of f on B_ϵ. Then we have the inequality $m_\epsilon \operatorname{vol}(B_\epsilon)\le\iiint_{B_\epsilon} f\,dV\le M_\epsilon\operatorname{vol}(B_\epsilon)$. Divide by $\operatorname{vol}(B_\epsilon)$, let $\epsilon\to 0$ and use continuity of f.

Review Exercises for Chapter 5

1. 81/2

3. $\frac{1}{4}e^2-e+\frac{9}{4}$

5. 81/2

7. $\frac{1}{4}e^2-e+\frac{9}{4}$

9. 7/60

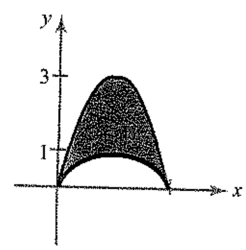
11. 1/2

13. In the notation of Figure 5.3.1,

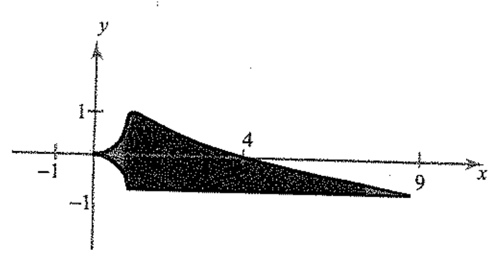
$$\iint_D dx\,dy=\int_a^b[\phi_2(x)-\phi_1(x)]\,dx.$$

15. (a) 0 (b) $\pi/24$ (c) 0

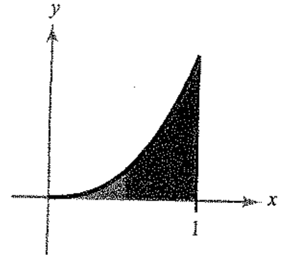
17. y-simple; $2\pi+\pi^2$.



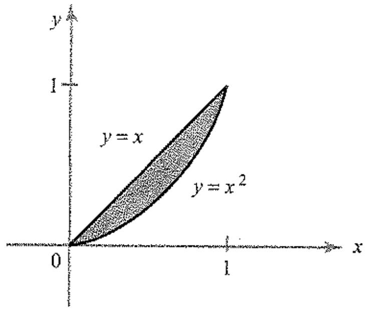
19. x-simple; 73/3.



21. y-simple; 33/140.



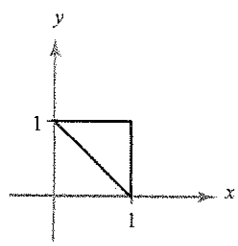
23. y-simple; $71/420$.



25. $1/3$

27. $19/3$

29. $7/12$



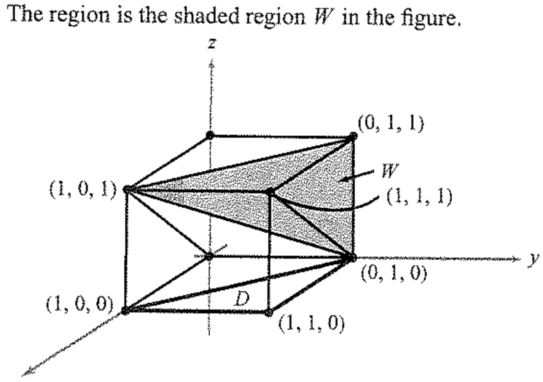
31. The function $f(x, y) = x^2 + y^2 + 1$ lies between 1 and $2^2 + 1 = 5$ on D, and so the integral lies between these values times 4π, the area of D.

33. Interchange the order of integration (the reader should draw a sketch in the (u, t) plane):

$$\int_0^x \int_0^t F(u)\,du\,dt = \int_0^x \int_u^x F(u)\,dt\,du = \int_0^x (x-u)F(u)\,du.$$

35. $\pi/12$

37. The region is the shaded region W in the figure.



The integral in the order $dy\,dx\,dz$, for example, is

$$\int_0^1 \int_z^1 \int_{1-x}^1 f(x, y, z)\,dy\,dx\,dz.$$

Chapter 6

Section 6.1

1. (a) One-to-one, Onto.
(b) Neither.
(c) One-to-one, Onto.
(d) Neither.

3. An appropriate linear function T is given by $T(x, y) = \left(x, -\frac{x}{3} + \frac{2y}{3}\right)$, or in matrix form, as:

$$T(\mathbf{v}) = A\mathbf{v} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \mathbf{v}.$$

5. $S =$ the unit disc minus its center.

7. $D = [0, 3] \times [0, 1]$; yes.

9. The image is the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$. T is not one-to-one, but becomes so if we eliminate the portion $x^* = 0$.

11. D is the set of (x, y, z) with $x^2 + y^2 + z^2 \le 1$ (the unit ball). T is not one-to-one, but is one-to-one on $(0, 1] \times (0, \pi) \times (0, 2\pi]$.

13. Showing that T is onto is equivalent in the 2×2 case to showing that the system $ax + by = e$, $cx + dy = f$ can always be solved for x and y, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

When you do this by elimination or by Cramer's rule, the quantity by which you must divide is det A. Thus, if det $A \ne 0$, the equations can always be solved.

15. Suppose that $T(\mathbf{x}) = T(\mathbf{y})$. Then

$$A\mathbf{x} + \mathbf{v} = A\mathbf{y} + \mathbf{v}$$
$$A\mathbf{x} = A\mathbf{y}.$$

By Exercise 12, this implies that $x = y$ if and only if det $A \ne 0$.
Showing that $T(\mathbf{x}) = A\mathbf{x} + \mathbf{v}$ is equivalent to showing that

$$T(\mathbf{x}) = A\mathbf{x} + \mathbf{v} = \mathbf{y}$$

or

$$A\mathbf{x} = \mathbf{y} - \mathbf{v}$$

has a solution for any choice of $\mathbf{y} \in \mathbb{R}^2$. This happens if and only if $\det A \neq 0$, by Exercise 13. Finally, verifying that T takes parallelograms to parallelograms follows exactly as in Exericse 14, by simply applying T to both sides of the given equation and simplifying.

17. We can show that T is not globally one-to-one by example. A simple choice is to compare the point $(1, 0)$ with $(-1, 0)$, which correspond to the polar coordinates $r = 1, \theta = 0$ and $r = 1, \theta = \pi$, respectively. We note:

$$\begin{aligned} T(1 \cos 0, 1 \sin 0) &= (1^2 \cos 0, 1^2 \sin 0) \\ &= (1^2 \cos 2\pi, 1^2 \sin 2\pi) = T(1 \cos \pi, 1 \sin \pi). \end{aligned}$$

Since $T(1, 0) = T(-1, 0)$, T is not one-to-one.

Section 6.2

1. A good substitution might be $u = 3x + 2y$, $v = x - y$, which has Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{5}$.

3. $\dfrac{\pi}{2}(e - 1)$

5. D is the region $0 \leq x \leq 4$, $\dfrac{1}{2}x + 3 \leq y \leq \dfrac{1}{2}x + 6$.
(a) 140 (b) −42

7. D^* is the region $0 \leq u \leq 1, 0 \leq v \leq 2$; $\dfrac{2}{3}(9 - 2\sqrt{2} - 3\sqrt{3})$.

9. π

11. $\dfrac{64\pi}{5}$

13. $3\pi/2$

15. $\dfrac{5\pi}{2}(e^4 - 1)$

17. $2a^2$

19. $\dfrac{21}{2}\left(e - \dfrac{1}{e}\right)$

21. $\dfrac{100\pi}{3}$

23. $4\pi[\sqrt{3}/2 - \log(1 + \sqrt{3}) + \log\sqrt{2}]$

25. $4\pi \log(a/b)$

27. 0

29. $2\pi[(b^2 + 1)e^{-b^2} - (a^2 + 1)e^{-a^2}]$

31. 24

33. (a) $\dfrac{4}{3}\pi abc$ (b) $\dfrac{4}{5}\pi abc$

35. (a) Check that if $T(u_1, v_1) = T(u_2, v_2)$, then $u_1 = u_2$ and $v_1 = v_2$.
(b) 160/3

37. $\dfrac{4}{9}a^{2/3} \displaystyle\iint_{D^*} [f((au^2)^{1/3}, (av^2)^{1/3})u^{-1/3}v^{-1/3}]\, du\, dv$

Section 6.3

1. $\left(1, \dfrac{1}{3}a\right)$

3. $[\pi^2 - \sin(\pi^2)]/\pi^3$

5. $\left(\dfrac{11}{18}, \dfrac{65}{126}\right)$

7. \$503.64

9. (a) δ, where δ is the (constant) mass density.
(b) 37/12

11. $500\pi\left(10 - \dfrac{1}{3}\right)$

13. $\left(\dfrac{1}{2}, \dfrac{1}{2}, \dfrac{1}{2}\right)$

15. 1/4

17. Letting δ be density, the moment of inertia is

$$\delta \int_0^k \int_0^{2\pi} \int_0^{a \sec\phi} (\rho^4 \sin^3 \phi)\, d\rho\, d\theta\, d\phi.$$

19. $(1.00 \times 10^8)m$

21. (a) The only plane of symmetry for the body of an automobile is the one dividing the left and right sides of the car.
(b) $\bar{z} \cdot \iiint_W \delta(x, y, z)\, dx\, dy\, dz$ is the z coordinate of the center of mass times the mass of W. Rearrangement of the formula for $\bar{z}$ gives the first line of the equation. The next step is justified by the additivity property of integrals. By symmetry, we can replace z by $-z$ and integrate in the region above the xy plane. Finally, we can factor the minus sign outside the second integral, and because $\delta(x, y, z) = \delta(u, v, -w)$, we are subtracting the second integral from itself. Thus, the answer is 0.

(c) In part (b), we showed that $\bar{z}$ times the mass of W is 0. Because the mass must be positive, $\bar{z}$ must be 0.

(d) By part (c), the center of mass must lie in both planes.

23. $V = -(4.71 \times 10^{19})Gm/R \approx -(3.04 \times 10^{9})m/R$, where m is the mass of a test particle at distance R from the planet's center.

25. In the x, y-plane, the circle D given by $(x-a)^2 + y^2 = r^2$ has center (and center of mass) $(a, 0)$. Also, the area of the circle has area $A(D) = \pi r^2$. Therefore, by Exercise 24 we have:

$$\mathrm{Vol}(W) = 2\pi(a)(\pi r^2).$$

Section 6.4

1. 4

3. 3/16

5. $\frac{1}{(1-\alpha)(1-\beta)}$

7. (a) 3π
(b) $\lambda < 1$

9. Integration of $\iint e^{-xy}\, dx\, dy$ with respect to x first and then y gives log 2. Reversing the order gives the integral on the left side of the equality stated in the exercise.

11. Integrate over $[\varepsilon, 1] \times [\varepsilon, 1]$ and let $\varepsilon \to 0$ to show the improper integral exists and equals 2 log 2.

13. $\frac{2\pi}{9}[(1+a^3)^{3/2} - a^{9/2} - 1]$

15. Use the fact that

$$\frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}} \le \frac{1}{\sqrt{1-x^2-y^2}}.$$

17. Use the fact that $e^{x^2+y^2}/(x-y) \ge 1/(x-y)$ on the given region.

19. Each integral equals 1/4, and Theorem 3 (Fubini's theorem) does apply.

21. Here, we let $D_1 = [0,1] \times [0,1]$, and $D_2 = [1,\infty] \times [1,\infty]$, as in the hint. On D_1, let $g(x,y) = \frac{1}{x^\alpha y^\beta}$ and $f(x,y) = \frac{1}{x^\alpha y^\beta + x^\gamma y^\rho}$. Since $x, y \ge 0$, it is clear that $0 \le f(x,y) \le g(x,y)$ for all points in D_1. Therefore, since $\iint_{D_1} g(x,y)\, dx\, dy$ exists by Exercise 5, we know that $\iint_{D_1} f(x,y)\, dx\, dy$ must also exist.

You may use a similar argument for the region D_2 by choosing a different $g(x,y)$ and applying the result of Exercise 6. Once $\iint f(x,y)\, dx\, dy$ exists over both the regions D_1 and D_2, it will exist also over their union $D = D_1 \cup D_2$.

Review Exercises for Chapter 6

1. (a) $T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u+v \\ 2v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
(b) $\iint_P f(x,y)\, dx\, dy = 4\iint_S f(2u+v, 2v)\, du\, dv$

3. 3 (Use the change of variables $u = x^2 - y^2$, $v = xy$.)

5. $\frac{1}{3}\pi(4\sqrt{2} - \frac{7}{2})$

7. $(5\pi/2)\sqrt{15}$

9. $abc/6$

11. Cut with the planes $x + y + z = \sqrt[3]{k/n}$, $1 \le k \le n-1$, k an integer.

13. $(25 + 10\sqrt{5})\pi/3$

15. $(e - e^{-1})/4$ (Use the change of variables $u = y - x$, $v = y + x$.)

17. $(9.92 \times 10^6)\pi$ grams

19. (a) 32
(b) This occurs at the point of the unit sphere $x^2 + y^2 + z^2 = 1$ inscribed in the cube.

21. $(0, 0, 3a^{4/8})$

23. $4\pi \ln(a/b)$

25. $\pi/2$

27. (a) $9/2$ (b) 64π

29. Work the integral with respect to y first on the region $D_{\varepsilon,L} = \{(x,y) | \varepsilon \le x \le L, 0 \le y \le x\}$ to obtain $I_{\varepsilon,L} = \iint_{D_{\varepsilon,L}} f\, dx\, dy = \int_\varepsilon^L x^{-3/2}(1 - e^{-x})\, dx$. The integrand is positive, and so $I_{\varepsilon,L}$ increases as $\varepsilon \to 0$ and $L \to \infty$. Bound $1 - e^{-x}$ above by x for $0 < x < 1$ and by 1 for $1 < x < \infty$ to see that $I_{\varepsilon,L}$ remains bounded and so must converge. The improper integral does exist.

31. (a) $1/6$ (b) $16\pi/3$

33. 2π

Chapter 7

Section 7.1

1. $\gamma(t) = \begin{cases} (3\cos \pi t, 3 \sin \pi t), & t \in [0, 1] \\ (6t - 9, 0), & t \in [1, 2] \end{cases}$

3. $\gamma(t) = \begin{cases} (t, \sin \pi t), & t \in [0, 1] \\ (2\pi - \pi t, 0), & t \in [1, 2] \end{cases}$

5. $\gamma(t) = (3 \cos 2\pi t, 4 \sin 2\pi t, 3), \quad t \in [0, 1]$

7. $\gamma(t) = (t, t, t^3), \quad t \in [-3, 2]$, or
$\gamma(t) = (5t - 3, 5t - 3, (5t - 3)^3), \quad t \in [0, 1]$

9.
$$\int_{\mathbf{c}} f(x, y, z)\, ds = \int_I f(x(t), y(t), z(t)) \|\mathbf{c}'(t)\|\, dt = \int_0^1 0 \cdot 1\, dt = 0$$

11. (a) 2 (b) $52\sqrt{14}$

13. $-\frac{1}{3}(1 + 1/e^2)^{3/2} + \frac{1}{3}(2^{3/2})$

15. (a) The path follows the straight line from (0, 0) to (1, 1) and back to (0, 0) in the xy plane. Over the path, the graph of f is a straight line from (0, 0, 0) to (1, 1, 1). The integral is the area of the resulting triangle covered twice and equals $\sqrt{2}$.

(b) $s(t) = \begin{cases} \sqrt{2}(1 - t^4) & \text{when } -1 \le t \le 0 \\ \sqrt{2}(1 + t^4) & \text{when } 0 < t \le 1. \end{cases}$

The path is

$$\mathbf{c}(s) = \begin{cases} (1 - s/\sqrt{2})(1, 1) & \text{when } 0 \le s \le \sqrt{2} \\ (s/(\sqrt{2} - 1))(1, 1) & \text{when } \sqrt{2} \le s \le 2\sqrt{2} \end{cases}$$

and $\int_{\mathbf{c}} f\, ds = \sqrt{2}$.

17. $2a/\pi$

19. (a) $[2\sqrt{5} + \log(2 + \sqrt{5})]/4$
(b) $(5\sqrt{5} - 1)/[6\sqrt{5} + 3 \log(2 + \sqrt{5})]$

21. Since the graph g is parameterized by $\gamma(t) = (t, g(t))$, we have $\gamma'(t) = (1, g'(t))$, and thus:

$$\|\gamma'(t)\| = \sqrt{1 + (g'(t))^2}.$$

23. 2

25. $\frac{\pi\sqrt{2}}{8}$

27. $\frac{\sqrt{2}}{3} t_0^3$

29. (a) $\sqrt{\frac{2}{g}}$

(b) Solving for y, we have:

$$y = -\sqrt{2x - x^2} + 1.$$

(Note that the negative square root was chosen for y.) Therefore our formula becomes:

$$\int_0^1 \frac{1}{-2g(\sqrt{2x - x^2} + 1)}\, dx.$$

Section 7.2

1. -1

3. (a) 3/2 (b) 0 (c) 0 (d) 147

5. 9

7. By the Cauchy–Schwarz inequality, $|\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)| \le \|\mathbf{F}(\mathbf{c}(t))\|\, \|\mathbf{c}'(t)\|$ for every t. Thus,

$$\left| \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \right| = \left| \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \right| \le \int_a^b |\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)|\, dt \le \int_a^b \|\mathbf{F}(\mathbf{c}(t))\|\, \|\mathbf{c}'(t)\|\, dt \le M \int_a^b \|\mathbf{c}'(t)\|\, dt = Ml.$$

9. $\frac{3}{4} - (n - 1)/(n + 1)$

11. 0

13. The length of $\mathbf{c}$.

15. If $\mathbf{c}'(t)$ is never 0, then the unit vector $\mathbf{T}(t) = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ is a continuous function of t and so is a smoothly turning tangent to the curve. The answer is no.

17. 7

19. Use the fact that $\mathbf{F}$ is a gradient to show that the work done is $\frac{1}{R_2} - \frac{1}{R_1}$, independent of the path.

21. (a) $\|\mathbf{c}'(x)\|$

(b) f has a positive derivative; it is one-to-one and onto $[0, L]$ by the mean-value and intermediate-value theorems. It has a differentiable inverse by the inverse function theorem.

(c) $g'(s) = 1/\|\mathbf{c}'(x)\|$, where $s = f(x)$.

(d) By the chain rule, $\mathbf{b}'(s) = \mathbf{c}'(x) \cdot g'(s)$, which has unit length by part (c).

Section 7.3

1. $z = 2(y - 1) + 1$

3. $18(z - 1) - 4(y + 2) - (x - 13) = 0$
or $18z - 4y - x - 13 = 0$.

5. Not regular when $u = 0$.

7. (a) (iii) (b) (i) (c) (ii) (d) (iv)

9. The vector $\mathbf{n} = (\cos v \sin u, \sin v \sin u, \cos u) = (x, y, z)$. The surface is the unit sphere centered at the origin.

11. $\mathbf{n} = -(\sin v)\mathbf{i} - (\cos v)\mathbf{k}$; the surface is a cylinder.

13. (a) $x = x_0 + (y - y_0)(\partial h/\partial y)(y_0, z_0) + (z - z_0)(\partial h/\partial z)(y_0, z_0)$ describes the plane tangent to $x = h(y, z)$ at (x_0, y_0, z_0), $x_0 = h(y_0, z_0)$.

(b) $y = y_0 + (x - x_0)(\partial k/\partial x)(x_0, z_0) + (z - z_0)(\partial k/\partial z)(x_0, z_0)$

15. $z - 6x - 8y + 3 = 0$

17. (a) The surface is a helicoid. It looks like a spiral ramp winding around the z axis. (See Figure 7.4.2.) It winds twice around, since θ goes up to 4π.

(b) $\mathbf{n} = \pm(1/\sqrt{1 + r^2})(\sin\theta, -\cos\theta, r)$

(c) $y_0 x - x_0 y + (x_0^2 + y_0^2)z = (x_0^2 + y_0^2)z_0$.

(d) If $(x_0, y_0, z_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0, \theta_0)$, then representing the line segment in the form $\{(r\cos\theta_0, r\sin\theta_0, \theta_0) | 0 \le r \le 1\}$ shows that the line lies in the surface. Representing the line as $\{(x_0, ty_0, z_0) | 0 \le t \le 1/(x_0^2 + y_0^2)\}$ and substituting into the results of part (c) shows that it lies in the tangent plane at (x_0, y_0, z_0).

19. (a) Using cylindrical coordinates leads to the parametrization

$$\Phi(z, \theta) = (\sqrt{25 + z^2}\cos\theta, \sqrt{25 + z^2}\sin\theta, z),$$
$$-\infty < z < \infty, 0 \le \theta \le 2\pi$$

as one possible solution.

(b) $\mathbf{n} = (\sqrt{25 + z^2}\cos\theta, \sqrt{25 + z^2}\sin\theta, -z)/\sqrt{25 + 2z^2}$

(c) $x_0 x + y_0 y = 25$

(d) Substitute the coordinates along these lines into the defining equation of the surface and the result of part (c).

21. (a) $u \mapsto u$, $v \mapsto v$, $u \mapsto u^3$, and $v \mapsto v^3$ all map $\mathbb{R}$ onto $\mathbb{R}$.

(b) $\mathbf{T}_u \times \mathbf{T}_v = (0, 0, 1)$ for Φ_1, and this is never $\mathbf{0}$. For the surface Φ_2, $\mathbf{T}_u \times \mathbf{T}_v = 9u^2v^2(0, 0, 1)$, and this is $\mathbf{0}$ along the u and v axes.

(c) We want to show that any two parametrizations of a surface that are smooth near a point will give the same tangent plane there. Thus, suppose $\Phi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ and $\Psi: B \subset \mathbb{R}^2 \to \mathbb{R}^3$ are parametrized surfaces such that

$$\Phi(u_0, v_0) = (x_0, y_0, z_0) = \Psi(s_0, t_0) \qquad \text{(i)}$$

and

$$\left(\mathbf{T}_u^\Phi \times \mathbf{T}_v^\Phi\right)\Big|_{(u_0, v_0)} \neq \mathbf{0}$$

$$\text{and} \qquad \left(\mathbf{T}_s^\Psi \times \mathbf{T}_t^\Psi\right)\Big|_{(s_0, t_0)} \neq \mathbf{0}, \qquad \text{(ii)}$$

so that Φ and Ψ are smooth and one-to-one in neighborhoods of (u_0, v_0) and (s_0, t_0), which we may as well assume are D and B. Suppose further that they "describe the same surface," that is, $\Phi(D) = \Phi(B)$. To see that they give the same tangent plane at (x_0, y_0, z_0), show that they have parallel normal vectors. To do this, show that there is an open set C with $(u_0, v_0) \in C \subset D$ and a differentiable map $f: C \to B$ such that $\Phi(u, v) = \Psi(f(u, v))$ for $(u, v) \in C$. Once you have done this, computation shows that the normal vectors are related by $\mathbf{T}_u^\Phi \times \mathbf{T}_v^\Phi = [\partial(s, t)/\partial(u, v)]\mathbf{T}_s^\Psi \times \mathbf{T}_t^\Psi$.

To see that there is such an f, notice that since $\mathbf{T}_s^\Psi \times \mathbf{T}_t^\Psi \neq \mathbf{0}$, at least one of the 2×2 determinants in the cross product is not zero. Assume, for example, that

$$\begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{vmatrix} \neq 0.$$

Now use the inverse function theorem to write (s, t) as a differentiable function of (x, y) in some neighborhood of (x_0, y_0).

(d) No.

23. (a) We plug the parametrization into the left hand side of the equation, and simplify:

$$\begin{aligned}
&(\sqrt{x^2 + y^2} - R)^2 + z^2 \\
&= (\sqrt{((R + r\cos u)\cos v)^2 + ((R + r\cos u)\sin v)^2} \\
&\quad - R)^2 + (r \sin u)^2 \\
&= (\sqrt{(R + r\cos u)^2} - R)^2 + r^2 \sin^2 u \\
&= (R + r\cos u - R)^2 + r^2 \sin^2 u \\
&= (r\cos u)^2 + r^2 \sin^2 u \\
&= r^2.
\end{aligned}$$

(b) We calculate the associated normal element

$$\begin{aligned} T_u \times T_v = (&-r\cos u\cos v(R + r\cos u),\\ &-r\cos u\sin v(R + r\cos u),\\ &-r\sin u(R + r\cos u)) \end{aligned}$$

and find that it is not equal to the zero vector for any choice of (u, v).

Section 7.4

1. 4π

3. $\frac{3}{2}\pi[\sqrt{2} + \log(1 + \sqrt{2})]$

5. (a) $(e^u \sin v, -e^u \cos v, e^u)$

(b) $x + z = \frac{\pi}{2}$

(c) $\pi\sqrt{2}(e - 1)$

7. $\frac{\sqrt{21}}{2}$

9. $\frac{1}{3}\pi(6\sqrt{6} - 8)$

11. The integral for the volume converges, whereas that for the area diverges.

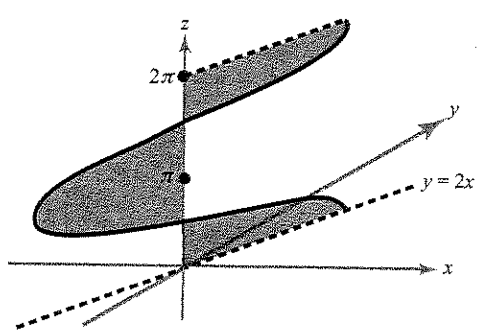
13. $A(E) = \int_0^{2\pi}\int_0^{\pi}$

$$\sqrt{a^2b^2\sin^2\phi\cos^2\phi + b^2c^2\sin^4\phi\cos^2\theta + a^2c^2\sin^4\phi\sin^2\theta}\, d\phi\, d\theta$$

15. $(\pi/6)(5\sqrt{5} - 1)$

17. $(\pi/2)\sqrt{6}$

19. $4\sqrt{5}$; for fixed θ, (x, y, z) moves along the horizontal line segment $y = 2x$, $z = \theta$ from the z axis out to a radius of $\sqrt{5}|\cos\theta|$ into quadrant 1 if $\cos\theta > 0$ and into quadrant 3 if $\cos\theta < 0$.



21. $(\pi + 2)/(\pi - 2)$

23. $\pi(a + b)\sqrt{1 + m^2}(b - a)$

25. $\frac{4}{15}(9\sqrt{3} - 8\sqrt{2} + 1)$

27. With $f(x, y) = \sqrt{R^2 - x^2 - y^2}$, (4) becomes

$$A(S') = \iint_D \sqrt{\frac{x^2 + y^2}{R^2 - x^2 - y^2} + 1}\, dx\, dy = \iint_D \frac{R}{\sqrt{R^2 - x^2 - y^2}}\, dx\, dy,$$

where D is the disc of radius R. Evaluate using polar coordinates, noting it is improper at the boundary, to get $2\pi R^2$.

Section 7.5

1. $\frac{512}{3}\sqrt{5}$

3. $11\sqrt{14}$

5. (a) For this surface parameterized by Φ, we have:

$$\begin{aligned} x^2 - y^2 &= (u + v)^2 - (u - v)^2\\ &= (u^2 + 2uv + v^2) - (u^2 - 2uv + v^2)\\ &= 4uv\\ &= 4z. \end{aligned}$$

(b) 0

7. $\frac{3\sqrt{2} + 5}{24}$

9. πa^3

11. (a) $\sqrt{2}\pi R^2$ (b) $2\pi R^2$

13. $\frac{\pi}{4}\left(\frac{5\sqrt{5}}{3} + \frac{1}{15}\right)$

15. $16\pi R^3/3$

17. (a) The sphere looks the same from all three axes, so these three integrals should be the same quantity with different labels on the axes.

(b) $4\pi R^4/3$

(c) $4\pi R^4/3$

19. 8

21. $(R/2, R/2, R/2)$

23. (a) Directly compute the vector cross product $\mathbf{T}_u \times \mathbf{T}_v$ and then calculate its length and compare your answer to the left-hand side.

(b) In this case, $F = 0$, so $A(s) = \iint_D \sqrt{EG}\, du\, dv$.

(c) $4\pi a^2$

25. Let $a = \partial x/\partial u$, $b = \partial y/\partial u$, $c = \partial x/\partial v$, and $d = \partial y/\partial v$. The conditions (a) and (b) in Exercise 16 are then $a^2 + b^2 = c^2 + d^2$ and $ac + bd = 0$. Show that $a \neq 0$ and, by a normalization argument, show that you can assume $a = 1$. Now calculate further.

27. $2a^2$

Section 7.6

1. $\dfrac{5\pi}{2}$

3. (a) 18π (b) 36π

5. $\pm 48\pi$ (the sign depends on orientation).

7. 4π

9. 2π (or -2π, if you choose a different orientation).

11. 2π

13. $12\pi/5$

15. With the usual spherical coordinate parametrization, $\mathbf{T}_\theta \times \mathbf{T}_\phi = -\sin\phi\, \mathbf{r}$ (see Example 1). Thus,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint \mathbf{F}\cdot(\mathbf{T}_\phi \times \mathbf{T}_\theta)\, d\phi\, d\theta = \iint (\mathbf{F}\cdot\mathbf{r}) \sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} F_r \sin\phi\, d\phi\, d\theta$$

and

$$\iint_S f\, dS = \int_0^{2\pi}\int_0^{\pi} f \sin\phi\, d\phi\, d\theta.$$

17. For a cylinder of radius $R = 1$ and normal component F_r,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_a^b \int_0^{2\pi} F_r\, d\theta\, dz.$$

19. $2\pi/3$

21. $\dfrac{2}{5}a^3 bc\pi$

Section 7.7

1. Apply formula (3) of this section and simplify; $H = 0$ and $K = -b^2/(u^2+b^2)^2$.

3. Apply formula (3) of this section and simplify.

5. $K = \dfrac{-4a^6b^6}{(a^4b^4 + 4b^4u^2 + 4a^4v^2)^2}$

7. Using the standard parametrization of the ellipsoid $\Phi(u, v) = (a\cos u \sin v, a \sin u \sin v, c\cos v)$, $u \in [0, 2\pi]$, $v \in [0, \pi]$, from Exercise 6 you should have found that the Gauss curvature of the ellipsoid is:

$$K = \frac{a^4c^2}{(a^4\cos^2 v + a^2c^2\cos^2 u \sin^2 v + a^2c^2 \sin^2 u \sin^2 v)^2} = \frac{a^4c^2}{(a^4\cos^2 v + a^2c^2\sin^2 v)^2}.$$

Then, the area area element for the ellipsoid is given as:

$$T_u \times T_v = \sin v\sqrt{a^4\cos^2 v + a^2c^2\sin^2 v}.$$

This yields the integral:

$$\int_0^{\pi}\int_0^{2\pi} \frac{a^4c^2\sin v}{(a^4\cos^2 v + a^2c^2\sin^2 v)^{\frac{3}{2}}}\, du\, dv.$$

To evaluate this integral, we try to get it into one of the standard forms found in the tables contained in the text:

$$\begin{aligned}\int_0^{\pi}\int_0^{2\pi} &\frac{a^4c^2\sin v}{(a^4\cos^2 v + a^2c^2\sin^2 v)^{\frac{3}{2}}}\, du\, dv \\ &= 2\pi\int_0^{\pi} \frac{a^4c^2\sin v}{a^3(a^2\cos^2 v + c^2\sin^2 v)^{\frac{3}{2}}}\, dv \\ &= 2\pi ac^2 \int_0^{\pi} \frac{\sin v}{(a^2\cos^2 v + c^2(1-\cos^2 v)^{\frac{3}{2}}}\, dv \\ &= 2\pi ac^2 \int_0^{\pi} \frac{\sin v}{((a^2-c^2)\cos^2 v + c^2)^{\frac{3}{2}}}\, dv \\ &= \frac{2\pi ac^2}{(a^2-c^2)^{\frac{3}{2}}}\int_0^{\pi} \frac{\sin v}{\left(\cos^2 v + \frac{c^2}{a^2-c^2}\right)^{\frac{3}{2}}}\, dv.\end{aligned}$$

At this point, make the substitution $w = \cos v$:

$$\frac{2\pi ac^2}{(a^2-c^2)^{\frac{3}{2}}}\int_0^{\pi} \frac{\sin v}{\left(\cos^2 v + \frac{c^2}{a^2-c^2}\right)^{\frac{3}{2}}}\, dv = \frac{2\pi ac^2}{(a^2-c^2)^{\frac{3}{2}}}\int_{-1}^{1} \frac{1}{\left((w)^2 + \left(\sqrt{\frac{c^2}{a^2-c^2}}\right)^2\right)^{\frac{3}{2}}}\, dw.$$

Finally, use the trigonometric substitution $w = \sqrt{\frac{c^2}{a^2-c^2}}\tan\theta$ to finish the integration. The final solution will simplify to 4π, verifying the Gauss–Bonnet theorem.

9. Apply formula (3) of this section and simplify.

11. Apply formula (2) of this section and simplify; $K = -h''/[(1+(h')^2)^2 h]$.

Review Exercises for Chapter 7

1. (a) $3\sqrt{2}(1-e^{6\pi})/13$
(b) $-\pi\sqrt{2}/2$
(c) $(236, 158\sqrt{26}-8)/35\cdot(25)^3$
(d) $8\sqrt{2}/189$

3. (a) $\frac{2}{\pi}+1$ (b) $-1/2$

5. $2a^3$

7. (a) A sphere of radius 5 centered at (2, 3, 0); $\mathbf{\Phi}(\theta,\phi) = (2+5\cos\theta\sin\phi, 3+5\sin\theta\sin\phi, 5\cos\phi)$; $0\le\theta\le 2\pi; 0\le\phi\le\pi$.
(b) An ellipsoid with center at (2, 0, 0); $\mathbf{\Phi}(\theta,\phi) = (2+(1/\sqrt{2})3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi)$; $0\le\theta\le 2\pi, 0\le\phi\le\pi$.
(c) An elliptic hyperboloid of one sheet; $\mathbf{\Phi}(\theta,z) = \left(\frac{1}{2}\sqrt{8+2z^2}\cos\theta, \frac{1}{3}\sqrt{8+2z^2}\sin\theta, z\right)$; $0\le\theta\le 2\pi, -\infty<z<\infty$.

9. $A(\mathbf{\Phi}) = \frac{1}{2}\int_0^{2\pi}\sqrt{3\cos^2\theta+5}\,d\theta$; $\mathbf{\Phi}$ describes the upper nappe of a cone with elliptical horizontal cross sections.

11. $11\sqrt{3}/6$

13. $\sqrt{2}/3$

15. $5\sqrt{5}/6$

17. (a) $(e^y\cos\pi z, xe^y\cos\pi z, -\pi xe^y\sin\pi z)$
(b) 0

19. $\frac{1}{2}(e^2+1)$

21. $\mathbf{n} = (1/\sqrt{5})(-1,0,2)$, $2z-x=1$

23. 0

25. If $\mathbf{F} = \nabla\phi$, then $\nabla\times\mathbf{F} = \mathbf{0}$ (at least if ϕ is of class C^2; see Theorem 1, Section 4.4). Theorem 3 of Section 7.2 shows that $\int_{\mathbf{c}}\nabla\phi\cdot d\mathbf{s} = 0$ because $\mathbf{c}$ is a closed curve.

27. (a) 24π (b) 24π (c) 60π

29. (a) $[\sqrt{R^2+p^2}(z_0-z_1)]/p$
(b) $\sqrt{2z_0(R^2+p^2)/p^2g}$

Chapter 8

Section 8.1

1. $\gamma(t) = \begin{cases}(2t-1, -t+1), & t\in[0,1]\\ (2t-1, 2t-2), & t\in[1,2]\\ (-4t+11, -t+4), & t\in[2,3]\end{cases}$

3. 8

5. 8

7. 61

9. -8

11. (a) 0
(b) $-\pi R^2$
(c) 0
(d) $-\pi R^2$

13. $3\pi a^2$

15. $3\pi/2$

17. $3\pi(b^2-a^2)/2$

19. (a) Both sides are 2π. (b) 0

21. 0

23. πab

25. A horizontal line segment divides the region into three regions of which Green's theorem applies; now use Exercise 16 or the technique in Figure 8.1.5.

27. $9\pi/8$

29. If $\varepsilon>0$, there is a $\delta>0$ such that $|u(\mathbf{q})-u(\mathbf{p})|<\varepsilon$ whenever $\|\mathbf{p}-\mathbf{q}\| = \rho<\delta$. Parametrize $\partial B_\rho(\mathbf{p})$ by $\mathbf{q}(\theta) = \mathbf{p}+\rho(\cos\theta,\sin\theta)$. Then

$$|I(\rho)-2\pi u(\mathbf{p})| \le \int_0^{2\pi}|u(\mathbf{q}(\theta))-u(\mathbf{p})|\,d\theta \le 2\pi.$$

31. If $\mathbf{p} = (p_1, p_2)$, parametrize $\partial B_\rho(\mathbf{p})$ by $\rho \mapsto (p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)$, then $I(\rho) = \int_0^{2\pi} u(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\, d\theta$. Differentiation under the integral sign gives

$$\begin{aligned}\frac{dI}{d\rho} &= \int_0^{2\pi} \nabla u \cdot (\cos\theta, \sin\theta)\, d\theta = \int_0^{2\pi} \nabla u \cdot \mathbf{n}\, d\theta \\ &= \frac{1}{\rho}\int_{\partial B_\rho} \frac{\partial u}{\partial \mathbf{n}}\, ds = \frac{1}{\rho}\iint_{B_\rho} \nabla^2 u\, dA\end{aligned}$$

(the last equality uses Exercise 30).

33. Using Exercise 32,

$$\begin{aligned}\iint_{B_R} u\, dA &= \int_0^R \int_0^{2\pi} u[\mathbf{p} + \rho(\cos\theta, \sin\theta)]\rho\, d\theta\, d\rho \\ &= \int_0^R \left(\int_{\partial B_\rho} u\, ds\right) d\rho \\ &= \int_0^R 2\pi\rho u(\mathbf{p})\, d\rho = \pi R^2 u(\mathbf{p}).\end{aligned}$$

35. Suppose u is subharmonic. We establish the assertions corresponding to Exercise 34(a) and (b). The argument for superharmonic functions is similar, with inequalities reversed.

Suppose $\nabla^2 u \ge 0$ and $u(\mathbf{p}) \ge u(\mathbf{q})$ for all $\mathbf{q}$ in $B_R(\mathbf{p})$. By Exercise 31, $I'(\rho) \ge 0$ for $0 < \rho \le R$, and so Exercise 32 shows that $2\pi u(\mathbf{p}) \le I(\rho) \le I(R)$ for $0 < \rho \le R$. If $u(\mathbf{q}) < u(\mathbf{p})$ for some $\mathbf{q} = \mathbf{p} + \rho(\cos\theta_0, \sin\theta_0) \in B_R(\mathbf{p})$, then, by continuity, there is an arc $[\theta_0 - \delta, \theta_0 + \delta]$ on $\partial B_\rho(\mathbf{p})$ where $u < u(\mathbf{p}) - d$ for some $d > 0$. This would mean that

$$\begin{aligned}2\pi u(\mathbf{p}) \le I(\rho) &= \frac{1}{\rho}\int_0^{2\pi} u[\mathbf{p} + \rho(\cos\theta, \sin\theta)]\rho\, d\theta \\ &\le (2\pi - 2\delta)u(\mathbf{p}) + 2\delta[u(\mathbf{p}) - d] \le 2\pi u(\mathbf{p}) - 2\,\delta d.\end{aligned}$$

This contradiction shows that we must have $u(\mathbf{q}) = u(\mathbf{p})$ for every $\mathbf{q}$ in $B_B(\mathbf{p})$.

If the maximum at $\mathbf{p}$ is absolute for D, the last paragraph shows that $u(\mathbf{x}) = u(\mathbf{p})$ for all $\mathbf{x}$ in some disc around $\mathbf{p}$. If $\mathbf{c}\colon [0, 1) \to D$ is a path from $\mathbf{p}$ to $\mathbf{q}$, then $u(\mathbf{c}(t)) = u(\mathbf{p})$ for all t in some interval $[0, b)$. Let b_0 be the largest $b \in [0, 1]$ such that $u(\mathbf{c}(t)) = u(\mathbf{p})$ for all $t \in [0, b)$. (Strictly speaking, this requires the notion of the least upper bound from a good calculus text.) Because u is continuous, $u(\mathbf{c}(b_0)) = u(\mathbf{p})$. If $b_0 \ne 1$, then the last paragraph would apply at $\mathbf{c}(b_0)$ and u is constantly equal to $u(\mathbf{p})$ on a disc around $\mathbf{c}(b_0)$. In particular, there is a $\delta > 0$ such that $u(\mathbf{c}(t)) = u(\mathbf{c}(b_0)) = u(\mathbf{p})$ on $[0, b_0 + \delta)$. This contradicts the maximality of b_0, so we must have $b_0 = 1$. That is, $\mathbf{c}(\mathbf{q}) = \mathbf{c}(\mathbf{p})$. Because $\mathbf{q}$ was an arbitrary point in D, u is constant on D.

37. Assume $\nabla^2 u_1 = 0$ and $\nabla^2 u_2 = 0$ are two solutions. Let $\phi = u_1 - u_2$. Then $\nabla^2\phi = 0$ and $\phi(x) = 0$ for all $x \in \partial D$. Consider the integral $\iint_D \phi\nabla^2\phi\, dA = -\iint_D \nabla\phi\cdot\nabla\phi\, dA$. Thus, $\iint_D \nabla\phi\cdot\nabla\phi\, dA = 0$, which implies that $\nabla\phi = \mathbf{0}$, and so ϕ is a constant function and hence must be identically zero.

Section 8.2

1. $$\gamma(t) = \begin{cases} (3t-1, 1, -6t+4), & t \in [0,1] \\ (2, 2t-1, -6t+4), & t \in [1,2] \\ (-3t+8, 3, 10t-28), & t \in [2,3] \\ (-1, -2t+9, 2t-4), & t \in [3,4] \end{cases}$$

$\Phi(u, v) = (u, v, 5 - 2u - 3v)$, $u \in [-1, 2]$, $v \in [1, 3]$

3. 0 (Note: $\mathbf{F}$ is a gradient field.)

5. π

7. 52

9. -2π

11. Each integral in Stokes' theorem is zero.

13. 0

15. $-4\pi/\sqrt{3}$

17. 0

19. $\pm 2\pi$

21. Using Faraday's law, $\iint_S [\nabla \times \mathbf{E} + \partial\mathbf{H}/\partial t]\cdot d\mathbf{S} = 0$ for any surface S. If the integrand were a nonzero vector at some point, then by continuity the integral over some small disc centered at that point and lying perpendicular to that vector would be nonzero.

23. The orientations of $\partial S_1 = \partial S_2$ must agree.

25. Suppose C is a closed loop on the surface drawn so that it divides the surface into two pieces, S_1 and S_2. For the surface of a doughnut (torus) you must use two closed loops; can you see why? Then C bounds both S_1 and S_2, but with positive orientation with respect to one and negative with respect to the other. Therefore,

$$\begin{aligned}\iint_S \nabla \times \mathbf{F}\cdot d\mathbf{S} &= \iint_{S_1} \nabla \times \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \nabla \times \mathbf{F}\cdot d\mathbf{S} \\ &= \int_C \mathbf{F}\cdot d\mathbf{s} - \int_C \mathbf{F}\cdot d\mathbf{s} = 0.\end{aligned}$$

27. (a) If $C = \partial S$, $\int_C \mathbf{v} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{s} = 0$.

(b) $\int_C \mathbf{v} \cdot d\mathbf{s} = \int_a^b \mathbf{v} \cdot \mathbf{c}'(t)\, dt = \mathbf{v} \cdot \int_a^b \mathbf{c}'(t)\, dt = \mathbf{v} \cdot (\mathbf{c}(b) - \mathbf{c}(a))$, where $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$ is a parametrization of C. (The vector integral is the vector whose components are the integrals of the component functions.) If C is closed, the last expression is 0.

29. Both integrals give $\pi/4$.

31. (a) 0 (b) π (c) π

33. -20π (or 20π if the opposite orientation is used).

35. One possible answer: The Möbius curve C is also the boundary of an *oriented* surface $\tilde{S}$; the equation in Faraday's law *is* valid for this new surface.

Section 8.3

1. (a) $f = x^2/2 + y^2/2 + C$

(b) $\mathbf{F}$ is not a gradient field.

(c) $f = \frac{1}{3}x^3 + xy^2 + C$

3. (a) There exists such a $\mathbf{G}$, but no such g.

(b) There exists such a g, but no such $\mathbf{G}$.

(c) There exists such a g, but no such $\mathbf{G}$.

(d) Neither function exists.

5. If $\mathbf{F} = \nabla f = \nabla g$ and C is a curve from $\mathbf{v}$ to $\mathbf{w}$, then $(f - g)(\mathbf{w}) - (f - g)(\mathbf{v}) = \int_C \nabla(f - g) \cdot d\mathbf{s} = 0$ and so $f - g$ is constant.

7. $x^2yz - \cos x + C$

9. Yes, it is the gradient of $g(x, y) = F(x) + F(y)$, where $F'(x) = f(x)$.

11. No; $\nabla \times \mathbf{F} = (0, 0, -x) \neq \mathbf{0}$.

13. $e \sin 1 + \frac{1}{3}e^3 - \frac{1}{3}$

15. 3.5×10^{29} ergs

17. (a) $f(x, y, z) = x^2yz$

(b) Not a gradient field.

(c) Not a gradient field.

(d) $f(x, y, z) = x^2 \cos y$

19. Use Theorem 7 in each case.

(a) $-3/2$ (b) -1

(c) $\cos(e^2) - \cos(1/e)/e$

21. (a) No.

(b) $\left(\frac{1}{2}z^2, xy - z, x^2y\right)$ or $\left(\frac{1}{2}z^2 - 2xyz - \frac{1}{2}y^2, -x^2z - z, 0\right)$.

23. $\frac{1}{3}(z^3\mathbf{i} + x^3\mathbf{j} + y^3\mathbf{k})$

25. $(-z \sin y + y \sin x, xz \cos y, 0)$ (Other answers are possible.)

27. (a) $\nabla \times \mathbf{F} = (0, 0, 2) \neq \mathbf{0}$

(b) Let $\mathbf{c}(t)$ be the path of an object in the fluid. Then $\mathbf{F}(\mathbf{c}(t)) = \mathbf{c}'(t)$. Let $\mathbf{c}(t) = (x(t), y(t), z(t))$. Then $x' = -y$, $y' = x$, and $z' = 0$, and so z is constant and the motion is parallel to the xy plane. Also, $x'' + x = 0$, $y'' + y = 0$. Thus, $x = A \cos t + B \sin t$ and $y = C \cos t + D \sin t$. Substituting these values in $x' = -y$, $y' = x$, we get $C = -B$, $D = A$, so that $x^2 + y^2 = A^2 + B^2$ and we have a circle.

(c) Counterclockwise.

29. (a)

$$\mathbf{F} = -\frac{GmM}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z);$$

$$\nabla \cdot \mathbf{F} = -GmM\left[\frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{x^2 + y^2 + z^2 - 3y^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{x^2 + y^2 + z^2 - 3z^2}{(x^2 + y^2 + z^2)^{5/2}}\right] = 0$$

(b) Let S be the unit sphere, S_1 the upper hemisphere, S_2 the lower hemisphere, and C the unit circle. If $\mathbf{F} = \nabla \times \mathbf{G}$, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{G} \cdot d\mathbf{s} - \int_C \mathbf{G} \cdot d\mathbf{s} = 0.$$

But $\iint_S \mathbf{F} \cdot d\mathbf{S} = -GmM \iint_S (\mathbf{r}/\|\mathbf{r}\|^3) \cdot \mathbf{n}\, dS = -4\pi GmM$, because $\|\mathbf{r}\| = 1$ and $\mathbf{r} = \mathbf{n}$ on S. Thus, $\mathbf{F} = \nabla \times \mathbf{G}$ is impossible. This does not contradict Theorem 8 because $\mathbf{F}$ is not smooth at the origin.

Section 8.4

1. 3

3. $4\pi r^3$

5. 4π

7. 3

9. (a) 0

(b) 4/15

(c) −4/15

11. 6

13. $\frac{7}{10}$

15. 1

17. Apply the divergence theorem to $f\mathbf{F}$ using $\nabla\cdot(f\mathbf{F}) = \nabla f\cdot\mathbf{F} + f\nabla\cdot\mathbf{F}$.

19. If $\mathbf{F} = \mathbf{r}/r^2$, then $\nabla\cdot\mathbf{F} = 1/r^2$. If $(0,0,0)\notin\Omega$, the result follows from Gauss' theorem. If $(0,0,0)\in\Omega$, we compute the integral by deleting a small ball $B_\varepsilon = \{(x,y,z) | (x^2+y^2+z^2)^{1/2} < \varepsilon\}$ around the origin and then letting $\varepsilon\to 0$:

$$\iiint_\Omega \frac{1}{r^2}\,dV = \operatorname*{limit}_{\varepsilon\to 0}\iiint_{\Omega\setminus B_\varepsilon}\frac{1}{r^2}\,dV = \operatorname*{limit}_{\varepsilon\to 0}\iint_{\partial(\Omega\setminus B_\varepsilon)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS$$

$$= \operatorname*{limit}_{\varepsilon\to 0}\left(\iint_{\partial\Omega}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS - \iint_{\partial B_\varepsilon}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS\right)$$

$$= \operatorname*{limit}_{\varepsilon\to 0}\left(\iint_{\partial\Omega}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS - 4\pi\varepsilon\right)$$

$$= \iint_{\partial\Omega}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS.$$

The integral over ∂B_ε is obtained from Theorem 10 (Gauss' law), because $r = \varepsilon$ everywhere on B_ε.

21. Use the vector identity for div($f\mathbf{F}$) and the divergence theorem for part (a). Use the vector identity $\nabla\cdot(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$ for part (b).

23. (a) If $\phi(\mathbf{p}) = \iiint_W \rho(\mathbf{q})/(4\pi\|\mathbf{p}-\mathbf{q}\|)\,dV(\mathbf{q})$, then

$$\nabla\phi(\mathbf{p}) = \iiint_W [\rho(\mathbf{q})/4\pi]\nabla_{\mathbf{p}}(1/\|\mathbf{p}-\mathbf{q}\|)\,dV(\mathbf{q})$$

$$= -\iiint_W [\rho(\mathbf{q})/4\pi][(\mathbf{p}-\mathbf{q})/\|\mathbf{p}-\mathbf{q}\|^3]\,dV(\mathbf{q}),$$

where $\nabla_{\mathbf{p}}$ means the gradient with respect to the coordinates of $\mathbf{p}$ and the integral is the vector whose components are the three component integrals. If $\mathbf{p}$ varies in $V\cup\partial V$ and $\mathbf{n}$ is the outward unit normal to ∂V, we can take the inner product using these components and collect the pieces as

$$\nabla\phi(\mathbf{p})\cdot\mathbf{n} = -\iiint_W \frac{\rho(\mathbf{q})}{4\pi}\frac{1}{\|\mathbf{p}-\mathbf{q}\|^3}(\mathbf{p}-\mathbf{q})\cdot\mathbf{n}\,dV(\mathbf{q}).$$

Thus,

$$\iint_{\partial V}\nabla\phi(\mathbf{p})\cdot\mathbf{n}\,dV(\mathbf{p}) = -\iint_{\partial V}\left(\iiint_W \frac{\rho(\mathbf{q})}{4\pi}\frac{1}{\|\mathbf{p}-\mathbf{q}\|^3}(\mathbf{p}-\mathbf{q})\cdot\mathbf{n}\,d\mathbf{q}\right)dV(\mathbf{p}).$$

There are essentially five variables of integration here, three placing $\mathbf{q}$ in W and two placing $\mathbf{p}$ on ∂V. Use Fubini's theorem to obtain

$$\iint_{\partial V}\nabla\phi\cdot\mathbf{n}\cdot d\mathbf{S}$$

$$= -\iiint_W \frac{\rho(\mathbf{q})}{4\pi}\left[\iint_{\partial V}\frac{(\mathbf{p}-\mathbf{q})\cdot\mathbf{n}}{\|\mathbf{p}-\mathbf{q}\|^3}\,dS(\mathbf{p})\right]dV(\mathbf{q}).$$

If V is a symmetric elementary region, Theorem 10 says that the inner integral is 4π if $\mathbf{q}\in V$ and 0 if $\mathbf{q}\notin V$. Thus,

$$\iint_{\partial V}\nabla\phi\cdot\mathbf{n}\,dS = -\iiint_{W\cap V}\rho(\mathbf{q})\,dV(\mathbf{q}).$$

Because $\rho = 0$ outside W,

$$\iint_{\partial V}\nabla\phi\cdot\mathbf{n}\,dS = -\iiint_V \rho(\mathbf{q})\,dV(\mathbf{q}).$$

If V is not a symmetric elementary region, subdivide it into a sum of such regions. The equation holds on each piece, and, upon adding them together, the boundary integrals along appropriately oriented interior boundaries cancel, leaving the desired result.

(b) By Theorem 9, $\iint_{\partial V}\nabla\phi\cdot d\mathbf{S} = \iiint_V \nabla^2\phi\,dV$, and so $\iiint_V \nabla^2\phi\,dV = -\iiint_V \rho\,dV$. Because both ρ and $\nabla^2\phi$ are continuous and this holds for arbitrarily small regions, we must have $\nabla^2\phi = -\rho$.

25. If the charge Q is spread evenly over the sphere S of radius R centered at the origin, the density of charge per unit area must be $Q/4\pi R^2$. If $\mathbf{p}$ is a point not on S and $\mathbf{q}\in S$, then the contribution to the electric field at $\mathbf{p}$ due to charge near $\mathbf{q}$ is directed along the vector $\mathbf{p}-\mathbf{q}$. Because the charge is evenly distributed, the tangential component of this contribution will be canceled by that from a symmetric point on the other side of the sphere at the same distance from $\mathbf{p}$. (Draw the picture.) The total resulting field must be radial. Because S looks the same from any point at a distance $\|\mathbf{p}\|$ from the origin, the field must depend only on radius and be of the form $\mathbf{E} = f(r)\mathbf{r}$.

If we look at the sphere Σ of radius $\|\mathbf{p}\|$, we have

$$(\text{charge inside } \Sigma) = \iint_{\Sigma} \mathbf{E}\cdot d\mathbf{S} = \iint_{\Sigma} f(\|\mathbf{p}\|)\mathbf{r}\cdot\mathbf{n}\, dS$$
$$= f(\|\mathbf{p}\|)\|\mathbf{p}\| \text{ area } \Sigma = 4\pi\|\mathbf{p}\|^3 f(\|\mathbf{p}\|).$$

If $\|\mathbf{p}\| < R$, there is no charge inside Σ; if $\|\mathbf{p}\| > R$, the charge inside Σ is Q, and so

$$\mathbf{E}(\mathbf{p}) = \begin{cases} \dfrac{1}{4\pi}\dfrac{Q}{\|\mathbf{p}\|^3}\mathbf{p} & \text{if } \|\mathbf{p}\| > R \\ 0 & \text{if } \|\mathbf{p}\| < R. \end{cases}$$

27. By Theorem 10, $\iint_{\partial M} \mathbf{F}\cdot d\mathbf{S} = 4\pi$ for any surface enclosing the origin. But if $\mathbf{F}$ were the curl of some field, then the integral over such a closed surface would have to be 0.

29. If $S = \partial W$, then $\iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iiint_W \nabla\cdot\mathbf{r}\, dV = \iiint_W 3\, dV = 3$ volume (W). For the geometric explanation, assume $(0,0,0)\in W$ and consider the skew cone with its vertex at $(0,0,0)$ with base ΔS and altitude $\|\mathbf{r}\|$. Its volume is $\frac{1}{3}(\Delta S)(\mathbf{r}\cdot\mathbf{n})$.

Section 8.5

1. (a) $(2xy^2 - yx^3)\, dx\, dy$

(b) $(x^2 + y^2)\, dx\, dy$

(c) $(x^2 + y^2 + z^2)\, dx\, dy\, dz$

(d) $(xy + x^2)\, dx\, dy\, dz$

(e) $dx\, dy\, dz$

3. (a) $2xy\, dx + (x^2 + 3y^2)\, dy$

(b) $-(x + y^2 \sin x)\, dx\, dy$

(c) $-(2x + y)\, dx\, dy$

(d) $dx\, dy\, dz$

(e) $2x\, dx\, dy\, dz$

(f) $2y\, dy\, dz - 2x\, dz\, dx$

(g) $-\dfrac{4xy}{(x^2+y^2)^2}\, dx\, dy$

(h) $2xy\, dx\, dy\, dz$

5. (a) $8\pi^2 + \dfrac{44\pi^3}{3} + \dfrac{11\pi^4}{2} + \dfrac{3\pi^5}{5}$

(b) $8\pi^2 + \dfrac{44\pi^3}{3} + \dfrac{53\pi^4}{2} + \dfrac{64\pi^5}{5} + \dfrac{7\pi^6}{3} + \dfrac{\pi^7}{7}$

(c) $8\pi + 10\pi^2 + 9\pi^3 + \dfrac{5\pi^4}{2} + \dfrac{\pi^5}{5}$

7. (a)

$$\begin{aligned}\text{Form}_2(\alpha\mathbf{V}_1 + \mathbf{V}_2) &= \text{Form}_2(\alpha A_1 + A_2, \alpha B_1 + B_2, \alpha C_1 + C_2) \\ &= (\alpha A_1 + A_2)\, dy\, dz + (\alpha B_1 + B_2)\, dz\, dx \\ &\quad + (\alpha C_1 + C_2)\, dx\, dy \\ &= \alpha(A_1\, dy\, dz + B_1\, dz\, dx + C_1\, dx\, dy) \\ &\quad + (A_2\, dy\, dz + B_2\, dz\, dx + C_2\, dx\, dy) \\ &= \alpha\,\text{Form}_2(\mathbf{V}_1) + \text{Form}_2(\mathbf{V}_2).\end{aligned}$$

(b)

$$d\omega = \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz\right)\wedge dx + A(dx)^2$$
$$+\left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy + \frac{\partial B}{\partial z}dz\right)\wedge dy + B(dy)^2$$
$$+\left(\frac{\partial C}{\partial x}dx + \frac{\partial C}{\partial y}dy + \frac{\partial C}{\partial z}dz\right)\wedge dz + C(dz)^2$$

But $(dx)^2 = (dy)^2 = (dz)^2 = dx\wedge dx = dy\wedge dy = dz\wedge dz = 0$, $dy\wedge dx = -dx\wedge dy$, $dz\wedge dy = -dy\wedge dz$, and $dx\wedge dz = -dz\wedge dx$. Hence,

$$d\omega = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right)dy\, dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial y}\right)dz\, dx$$
$$+\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)dx\, dy$$
$$= \text{Form}_2(\text{curl}\,\mathbf{V}).$$

9. An oriented 1-manifold is a curve. Its boundary is a pair of points that may be considered a 0-manifold. Therefore, ω is a 0-form or function, and $\int_{\partial M} d\omega = \omega(b) - \omega(a)$ if the curve M runs from a to b. Furthermore, $d\omega$ is the 1-form $(\partial\omega/\partial x)\, dx + (\partial\omega/\partial y)\, dy$. Therefore, $\int_M d\omega$ is the line integral $\int_M (\partial\omega/\partial x)\, d\omega + (\partial\omega/dy)\, dy = \int_M \nabla\omega\cdot d\mathbf{s}$. Thus, we obtain Theorem 3 of Section 7.2, $\int_M \nabla\omega\cdot d\mathbf{s} = \omega(b) - \omega(a)$.

11. Put $\omega = F_1\, dx\, dy + F_2\, dy\, dz + F_3\, dz\, dx$. The integral becomes

$$\iint_{\partial T}\omega = \iiint_T d\omega$$
$$= \iiint_T\left(\frac{\partial F_1}{\partial z} + \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial y}\right)dx\, dy\, dz.$$

(a) 0

(b) 40

13. Consider $\omega = x\, dy\, dz + y\, dz\, dx + z\, dx\, dy$. Compute that $d\omega = 3\, dx\, dy\, dz$, so that $\frac{1}{3}\iint_{\partial R}\omega = \frac{1}{3}\iiint_R d\omega = \iiint_R dx\, dy\, dz = v(R)$.

Review Exercises for Chapter 8

1. (a) $2\pi a^2$
(b) 0

3. 0

5. (a) $f = x^4/4 - x^2y^3$
(b) $-1/4$

7. (a) Check that $\nabla \times \mathbf{F} = \mathbf{0}$.
(b) $f = 3x^2 y \cos z + C$
(c) 0

9. $23/6$

11. No: $\nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$.

13. (a) $\nabla f = 3ye^{z^2}\mathbf{i} + 3xe^{z^2}\mathbf{j} + 6xyze^{z^2}\mathbf{k}$
(b) 0
(c) Both sides are 0.

15. $8\pi/3$

17. $\pi a^2/4$

19. 21

21. (a) $\mathbf{G}$ is conservative; $\mathbf{F}$ is not.
(b) $\mathbf{G} = \nabla\phi$ if $\phi = (x^4/4)+(y^4/4)-\frac{3}{2}x^2y^2+\frac{1}{2}z^2+C$, where C is any constant.
(c) $\int_\alpha \mathbf{F}\cdot d\mathbf{s} = 0$; $\int_\alpha \mathbf{G}\cdot d\mathbf{s} = -\frac{1}{2}$;
$\int_\beta \mathbf{F}\cdot d\mathbf{s} = \frac{1}{3}$; $\int_\beta \mathbf{G}\cdot d\mathbf{s} = -\frac{1}{2}$

23. Use $(\nabla\cdot\mathbf{F})(x_0, y_0, z_0) = \text{limit}_{\rho\to 0}\frac{1}{V(\Omega_\rho)}\iint_{\partial\Omega_\rho} \mathbf{F}\cdot\mathbf{n}\,dS$ from Section 8.4.