example 4 Let $\mathbf{F} = y^3\mathbf{i} + x^5\mathbf{j}$. Compute the integral of the normal component of $\mathbf{F}$ around the unit square.

solution This can be done using the divergence theorem. Indeed

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}\, dA.$$

But div $\mathbf{F} = 0$, and so the integral is zero. ▲

exercises

1. Let D be the triangle in the xy plane with vertices at $(-1, 1)$, $(1, 0)$, and $(3, 2)$. Describe the boundary ∂D as a piecewise smooth curve, oriented counterclockwise.

2. Let D be the region in the xy plane lying between the curves $y = x^2 + 4$ and $y = 2x^2$. Describe the boundary ∂D as a piecewise smooth curve, oriented counterclockwise.

In Exercises 3 to 6, verify Green's theorem for the indicated region D and boundary ∂D, and functions P and Q.

3. $D = [-1, 1] \times [-1, 1]$, $P(x, y) = -y$, $Q(x, y) = x$

4. $D = [-1, 1] \times [-1, 1]$, $P(x, y) = x$, $Q(x, y) = y$

5. $D = [-1, 1] \times [-1, 1]$, $P(x, y) = x - y$, $Q(x, y) = x + y$
[HINT: Use 3 and 4.]

6. $D = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$, $P(x, y) = \sin x$, $Q(x, y) = \cos y$

7. Let C be the closed, piecewise smooth curve formed by traveling in straight lines between the points $(-2, 1)$, $(-2, -3)$, $(1, -1)$, $(1, 5)$, and back to $(-2, 1)$, in that order. Use Green's theorem to evaluate the integral:

$$\int_C (2xy)\, dx + (xy^2)\, dy$$

8. A particle travels across a flat surface, moving due east for 3 m, then due north for 4 m, and then returns to its origin. A force field acts on the particle, given by $\mathbf{F}(x, y) = (3x + 4y^2)\mathbf{i} + (10xy)\mathbf{j}$. (Here we assume that $\mathbf{j}$ points north.) Use Green's theorem to find the work done on the particle by $\mathbf{F}$.

9. Evaluate $\int_C y\, dx - x\, dy$, where C is the boundary of the square $[-1, 1] \times [-1, 1]$ oriented in the counterclockwise direction, using Green's theorem.

10. Find the area of the disc D of radius R using Green's theorem.

11. Verify Green's theorem for the disc D with center $(0, 0)$ and radius R and the functions:

(a) $P(x, y) = xy^2$, $Q(x, y) = -yx^2$

(b) $P(x, y) = x + y$, $Q(x, y) = y$

(c) $P(x, y) = xy = Q(x, y)$

(d) $P(x, y) = 2y$, $Q(x, y) = x$

12. Using the divergence theorem, show that $\int_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = 0$, where $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$ and D is the unit disc. Verify this directly.

13. Find the area bounded by one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, where $a > 0$, and $0 \le \theta \le 2\pi$, and the x axis (use Green's theorem).

14. Under the conditions of Green's theorem, prove that

(a) $$\int_{\partial D} PQ\, dx + PQ\, dy = \iint_D \left[Q\left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right) + P\left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right)\right] dx\, dy$$

(b) $$\int_{\partial D} \left(Q\frac{\partial P}{\partial x} - P\frac{\partial Q}{\partial x}\right) dx + \left(P\frac{\partial Q}{\partial y} - Q\frac{\partial P}{\partial y}\right) dy = 2\iint_D \left(P\frac{\partial^2 Q}{\partial x\, \partial y} - Q\frac{\partial^2 P}{\partial x\, \partial y}\right) dx\, dy$$

15. Evaluate the line integral

$$\int_C (2x^3 - y^3)\, dx + (x^3 + y^3)\, dy,$$

where C is the unit circle, and verify Green's theorem for this case.

16. Prove the following generalization of Green's theorem: Let D be a region in the xy plane with boundary a finite number of oriented simple closed curves. Suppose that by means of a finite number of line segments parallel to the coordinate axes, D can be decomposed into a finite number of simple regions D_i with the boundary of each D_i oriented counterclockwise (see Figure 8.1.5). Then if P and Q are of class C^1 on D,

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial D} P\,dx + Q\,dy,$$

where ∂D is the oriented boundary of D. (HINT: Apply Green's theorem to each D_i.)

17. Verify Green's theorem for the integrand of Exercise 15 (that is, with $P = 2x^3 - y^3$ and $Q = x^3 + y^3$) and the annular region D described by $a \le x^2 + y^2 \le b$, with boundaries oriented as in Figure 8.1.5.

18. Let D be a region for which Green's theorem holds. Suppose f is harmonic; that is,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

on D. Prove that

$$\int_{\partial D} \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = 0.$$

19. (a) Verify the divergence theorem for $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and D the unit disc $x^2 + y^2 \le 1$.

(b) Evaluate the integral of the normal component of $2xy\mathbf{i} - y^2\mathbf{j}$ around the ellipse defined by $x^2/a^2 + y^2/b^2 = 1$.

20. Let $P(x, y) = -y/(x^2 + y^2)$ and $Q(x, y) = x/(x^2 + y^2)$. Assuming D is the unit disc, investigate why Green's theorem fails for this P and Q.

21. Use Green's theorem to evaluate $\int_{C^+} (y^2 + x^3)\,dx + x^4\,dy$, where C^+ is the perimeter of the square $[0, 1] \times [0, 1]$ in the counterclockwise direction.

22. Verify Theorem 3 by showing that $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \partial Q/\partial x - \partial P/\partial y$.

23. Use Theorem 2 to compute the area inside the ellipse $x^2/a^2 + y^2/b^2 = 1$.

24. Use Theorem 2 to recover the formula $A = \frac{1}{2}\int_a^b r^2\,d\theta$ for a region in polar coordinates.

25. Sketch the proof of Green's theorem for the region shown in Figure 8.1.10.

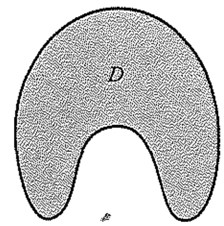

figure 8.1.10 Prove Green's theorem for this region.

26. Prove the identity

$$\int_{\partial D} \phi \nabla\phi \cdot \mathbf{n}\,ds = \iint_D (\phi\nabla^2\phi + \nabla\phi \cdot \nabla\phi)\,dA.$$

27. Use Green's theorem to find the area of one loop of the four-leafed rose $r = 3\sin 2\theta$. (HINT: $x\,dy - y\,dx = r^2 d\theta$.)

28. Show that if C is a simple closed curve that bounds a region to which Green's theorem applies, then the area of the region D bounded by C is

$$A = \int_{\partial D} x\,dy = -\int_{\partial D} y\,dx.$$

Show how this implies Theorem 2.

Exercises 29 to 37 illustrate the application of Green's theorem to partial differential equations. (Further applications are given in the Internet supplement.) They are particularly concerned with solutions to Laplace's equation, that is, with harmonic functions. For these exercises, let D be an open region in $\mathbb{R}^2$ with boundary ∂D. Let u: $D \cup \partial D \to \mathbb{R}$ be a continuous function that is of class C^2 on D. Suppose $\mathbf{p} \in D$ and the closed discs $B_\rho = B_\rho(\mathbf{p})$ of radius ρ centered at $\mathbf{p}$ are contained in D for $0 < \rho \le R$. Define $I(\rho)$ by

$$I(\rho) = \frac{1}{\rho}\int_{\partial B_\rho} u\,ds.$$

29. Show that $\text{limit}_{\rho\to 0}\, I(\rho) = 2\pi u(\mathbf{p})$.

30. Let $\mathbf{n}$ denote the outward unit normal to ∂B_ρ and $\partial u/\partial n = \nabla u \cdot \mathbf{n}$. Show that

$$\int_{\partial B_\rho} \frac{\partial u}{\partial n}\,ds = \iint_{B_\rho} \nabla^2 u\,dA.$$

31. Using Exercise 30, show that $I'(\rho) = (1/\rho)\iint_{B_\rho} \nabla^2 u\, dA$.

32. Suppose u satisfies Laplace's equation: $\nabla^2 u = 0$ on D. Use the preceding exercises to show that

$$u(\mathbf{p}) = \frac{1}{2\pi R}\int_{\partial B_R} u\, ds.$$

(This expresses the fact that the value of a harmonic function at a point is the average of its values on the circumference of any disc centered about it.)

33. Use Exercise 32 to show that if u is harmonic (i.e., if $\nabla^2 u = 0$), then $u(\mathbf{p})$ can be expressed as an area integral

$$u(\mathbf{p}) = \frac{1}{\pi R^2}\iint_{B_R} u\, dA.$$

34. Suppose u is a harmonic function defined on D (i.e., $\nabla^2 u = 0$ on D) and that u has a local maximum (or minimum) at a point $\mathbf{p}$ in D.

(a) Show that u must be constant on some disc centered at $\mathbf{p}$. (HINT: Use the results of Exercise 25.)

(b) Suppose that D is path-connected [i.e., for any points $\mathbf{p}$ and $\mathbf{q}$ in D, there is a continuous path $\mathbf{c}\colon [0, 1] \to D$ such that $\mathbf{c}(0) = \mathbf{p}$ and $\mathbf{c}(1) = \mathbf{q}$] and that for some $\mathbf{p}$ the maximum or minimum at $\mathbf{p}$ is absolute; thus, $u(\mathbf{q}) \le u(\mathbf{p})$ or $u(\mathbf{q}) \ge u(\mathbf{p})$ for every $\mathbf{q}$ in D. Show that u must be constant on D.

(The result in this Exercise is called a ***strong maximum*** or ***minimum principle*** for harmonic functions. Compare this with Exercises 46 to 50 in Section 3.3.)

35. A function is said to be ***subharmonic*** on D if $\nabla^2 u \ge 0$ everywhere in D. It is said to be ***superharmonic*** if $\nabla^2 u \le 0$.

(a) Derive a strong maximum principle for subharmonic functions.

(b) Derive a strong minimum principle for superharmonic functions.

36. Suppose D is the disc $\{(x, y) \mid x^2 + y^2 < 1\}$ and C is the circle $\{(x, y) \mid x^2 + y^2 = 1\}$. In the Internet supplement, we shall show that if f is a continuous real-valued function on C, then there is a continuous function u on $D \cup C$ that agrees with f on C and is harmonic on D. That is, f has a harmonic extension to the disc. Assuming this, show the following:

(a) If q is a nonconstant continuous function on $D \cup C$ that is subharmonic (but not harmonic) on D, then there is a continuous function u on $D \cup C$ that is harmonic on D such that u agrees with q on C and $q < u$ everywhere on D.

(b) The same assertion holds if "subharmonic" is replaced by "superharmonic" and "$q < u$" by "$q > u$."

37. Let D be as in Exercise 36. Let $f\colon D \to \mathbb{R}$ be continuous. Show that a solution to the equation $\nabla^2 u = 0$ satisfying $u(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in \partial D$ is unique.

38. Use Green's theorem to prove the change of variables formula in the following special case:

$$\iint_D dx\, dy = \iint_{D^*} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

for a transformation $(u, v) \mapsto (x(u, v), y(u, v))$.

8.2 Stokes' Theorem

Stokes' theorem relates the line integral of a vector field around a simple closed curve C in $\mathbb{R}^3$ to an integral over a surface S for which C is the boundary. In this regard it is very much like Green's theorem.

Stokes' Theorem for Graphs

Let us begin by recalling a few facts from Chapter 7. Consider a surface S that is the graph of a function $f(x, y)$, so that S is parametrized by

$$\begin{cases} x = u \\ y = v \\ z = f(u, v) = f(x, y) \end{cases}$$