example 4 Show that $f(x, y) = 1/\sqrt{x^2 + y^2}$ is integrable over the unit disk D and evaluate $\iint_D f\,dA$.

solution Let D_δ be the disk of radius δ centered at the origin. Then f is continuous everywhere on D except at $(0, 0)$. Thus, $\iint_{D\setminus D_\delta} f\,dA$ exists. To evaluate this integral, we change variables to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$. Then $f(r\cos\theta, r\sin\theta) = 1/r$, and

$$\iint_{D\setminus D_\delta} f\,dA = \int_\delta^1 \int_0^{2\pi} \frac{1}{r} f\,d\theta\,dr = \int_\delta^1 \int_0^{2\pi} d\theta\,dr = 2\pi(1 - \delta).$$

Thus,

$$\iint_D f\,dA = \lim_{\delta\to 0} \iint_{D\setminus D_\delta} f\,dA = 2\pi. \quad \blacktriangle$$

More generally, we can, in an analogous manner, define the integral of nonnegative functions f that are continuous except at a finite number of points in D. We can also combine both types of improper integrals; that is, we may consider functions that are continuous except at a finite number of points on D or at points on the boundary of D, and define $\iint_D f\,dA$ appropriately.

If f takes both positive and negative values, we can use a more advanced integration theory, called the *Lebesgue integral*, to generalize the notion of convergent integral $\iint_D f\,dA$. Using this theory, it is possible to show that if $\iint_D f\,dA$ exists, it can then be evaluated as an iterated integral. This latter fact is also known as Fubini's theorem.

Unbounded Regions

As was mentioned previously, we will leave consideration of unbounded regions to the exercise section. However, we must point out that we have already addressed the main idea in Example 5 of Section 6.2 on the Gaussian integral. In that example, we integrated $\exp(-x^2 - y^2)$ over all of $\mathbb{R}^2$ by integrating first over a disk of radius a and then letting $a \to \infty$.

exercises

In Exercises 1 to 4, evaluate the following integrals if they exist (discuss how you define the integral if it was not given in the text).

1. $\displaystyle\iint_D \frac{1}{\sqrt{xy}}\,dA$, where $D = [0, 1] \times [0, 1]$

2. $\displaystyle\iint_D \frac{1}{\sqrt{|x - y|}}\,dx\,dy$, where $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1, y \le x\}$

3. $\displaystyle\iint_D (y/x)\,dx\,dy$, where D is bounded by $x = 1$, $x = y$, and $x = 2y$

4. $\displaystyle\int_0^1 \int_0^{e^y} \log x\,dx\,dy$

5. Let $D = [0, 1] \times [0, 1]$. Let $0 < \alpha < 1$ and $0 < \beta < 1$. Evaluate:

$$\iint_D \frac{dx\,dy}{x^\alpha y^\beta}.$$

6. Let $D = [1, \infty) \times [1, \infty]$. Let $1 < \gamma$ and $1 < \rho$. Evaluate:

$$\iint_D \frac{dx\,dy}{x^\gamma y^\rho}.$$

7. (a) Evaluate

$$\iint_D \frac{dA}{(x^2+y^2)^{2/3}},$$

where D is the unit disk in $\mathbb{R}^2$.

(b) Determine the real numbers λ for which the integral

$$\iint_D \frac{dA}{(x^2+y^2)^{\lambda}}$$

is convergent, where again D is the unit disk.

8. (a) Discuss how you would define $\iint_D f\, dA$ if D is an unbounded region—for example, the set of (x, y) such that $a \le x < \infty$ and $\phi_1(x) \le y \le \phi_2(x)$, where $\phi_1 \le \phi_2$ are given (Figure 6.4.5).

(b) Evaluate $\iint_D xye^{-(x^2+y^2)} dx\, dy$ if $x \ge 0$, $0 \le y \le 1$.

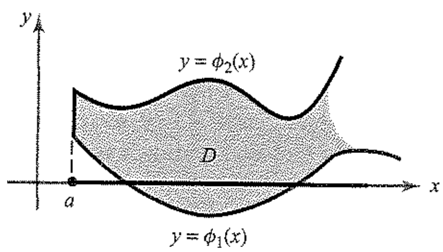

figure **6.4.5** An unbounded region D.

9. Using Exercise 8, integrate e^{-xy} for $x \ge 0$, $1 \le y \le 2$ in two ways. Assuming Fubini's theorem can be used, show that

$$\int_0^{\infty} \frac{e^{-x}-e^{-2x}}{x}\, dx = \log 2.$$

10. Show that the integral

$$\int_0^1 \int_0^a (x/\sqrt{a^2-y^2})\, dy\, dx$$

exists, and compute its value.

11. Discuss whether the integral

$$\iint_D \frac{x+y}{x^2+2xy+y^2}\, dx\, dy$$

exists where $D = [0, 1] \times [0, 1]$. If it exists, compute its value.

12. We can also consider improper integrals of functions that fail to be continuous on entire curves lying in some region D. For example, by breaking $D = [0, 1] \times [0, 1]$ into two regions, define and then discuss the convergence of the integral

$$\iint_D \frac{1}{\sqrt{|x-y|}}\, dx\, dy.$$

13. Let W be the first octant of the ball $x^2+y^2+z^2 \le a^2$, where $x \ge 0$, $y \ge 0$, $z \ge 0$. Evaluate the improper integral

$$\iiint_W \frac{(x^2+y^2+z^2)^{1/4}}{\sqrt{z+(x^2+y^2+z^2)^2}}\, dx\, dy\, dz$$

by changing variables.

14. Let f be a nonnegative function that may be unbounded and discontinuous on the boundary of an elementary region D. Let g be a similar function such that $f(x, y) \le g(x, y)$ whenever both are defined. Suppose $\iint_D g(x, y)\, dA$ exists. Argue informally that this implies the existence of $\iint_D f(x, y)\, dA$.

15. Use Exercise 14 to show that

$$\iint_D \frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}}\, dy\, dx$$

exists where D is the unit disk $x^2+y^2 \le 1$.

16. Let f be as in Exercise 14 and let g be a function such that $0 \le g(x, y) \le f(x, y)$ whenever both are defined. Suppose that $\iint_D g(x, y)\, dA$ does not exist. Argue informally that $\iint_D f(x, y)\, dA$ cannot exist.

17. Use Exercise 16 to show that

$$\iint_D \frac{e^{x^2+y^2}}{x-y}\, dy\, dx$$

does not exist, where D is the set of (x, y) with $0 \le x \le 1$ and $0 \le y \le x$.

18. Let D be the unbounded region defined as the set of (x, y, z) with $x^2+y^2+z^2 \ge 1$. By making a change of variables, evaluate the improper integral

$$\iiint_D \frac{dx\, dy\, dz}{(x^2+y^2+z^2)^2}.$$

19. Evaluate

$$\int_0^1 \int_0^y \frac{x}{y}\, dx\, dy \quad \text{and} \quad \int_0^1 \int_x^1 \frac{x}{y}\, dy\, dx.$$

Does Fubini's theorem apply?

20. In Exercise 17 of Section 5.2 we showed that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dy\,dx \neq \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dx\,dy.$$

Thus, Fubini's theorem does not hold here, even though the iterated improper integrals both exist. What went wrong?

21. If $0 \le f(x, y) \le g(x, y)$ for all $(x, y) \in D$, and the improper integral of g

$$\iint_D g(x, y)\,dx\,dy$$

exists, then $\iint_D f(x, y)\,dx\,dy$ also exists. Use this fact and exercises 5 and 6 to argue that if $0 < \alpha, \beta < 1$ and $1 < \gamma, \rho$, then

$$\iint_D \frac{dx\,dy}{x^\alpha y^\beta + x^\gamma y^\rho}$$

exists, where $D = [0, \infty) \times [0, \infty)$.
[HINT: Write $D = D_1 \cup D_2$ and apply Exercise 14 to each D_i separately.]

review exercises for chapter 6

1. (a) Find a linear transformation taking the square $S = [0, 1] \times [0, 1]$ to the parallelogram P with vertices $(0, 0)$, $(2, 0)$, $(1, 2)$, $(3, 2)$.

(b) Write down a change of variables formula appropriate to the transformation you found in part (a).

2. (a) Find the image of the square $[0, 1] \times [0, 1]$ under the transformation $T(x, y) = (2x, x + 3y)$.

(b) Write down a change of variables formula appropriate to the transformation and the region you found in part (a).

3. Let B be the region in the first quadrant bounded by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 4$. Evaluate $\iint_B (x^2 + y^2)\,dx\,dy$ using the change of variables $u = x^2 - y^2$, $v = xy$.

4. In parts (a) to (d), make the indicated change of variables. (Do not evaluate.)

(a) $\displaystyle\int_0^1 \int_{-1}^1 \int_{-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} (x^2 + y^2)^{1/2} dx\,dy\,dz$, cylindrical coordinates

(b) $\displaystyle\int_{-1}^1 \int_{-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} \int_{-\sqrt{(4-x^2-y^2)}}^{\sqrt{(4-x^2-y^2)}} xyz\,dz\,dx\,dy$, cylindrical coordinates

(c) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(2-y^2)}}^{\sqrt{(2-y^2)}} \int_{\sqrt{(x^2+y^2)}}^{\sqrt{(4-x^2-y^2)}} z^2 dz\,dx\,dy$, spherical coordinates

(d) $\displaystyle\int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^3 \sin 2\phi\,d\theta\,d\phi\,d\rho$, rectangular coordinates

5. Find the volume inside the surfaces $x^2 + y^2 = z$ and $x^2 + y^2 + z^2 = 2$.

6. Find the volume enclosed by the cone $x^2 + y^2 = z^2$ and the plane $2z - y - 2 = 0$.

7. A cylindrical hole of diameter 1 is bored through a sphere of radius 2. Assuming that the axis of the cylinder passes through the center of the sphere, find the volume of the solid that remains.

8. Let C_1 and C_2 be two cylinders of infinite extent, of diameter 2, and with axes on the x and y axes, respectively. Find the volume of their intersection, $C_1 \cap C_2$.

9. Find the volume bounded by $x/a + y/b + z/c = 1$ and the coordinate planes.

10. Find the volume determined by $z \le 6 - x^2 - y^2$ and $z \ge \sqrt{x^2 + y^2}$.

11. The *tetrahedron* defined by $x \ge 0, y \ge 0, z \ge 0, x + y + z \le 1$ is to be sliced into n segments of equal volume by planes parallel to the plane $x + y + z = 1$. Where should the slices be made?

12. Let E be the solid ellipsoid $E = \{(x, y, z) \mid (x^2/a^2) + (y^2/b^2) + (z^2/c^2) \le 1\}$, where $a > 0$, $b > 0$, and $c > 0$. Evaluate

$$\iiint xyz\,dx\,dy\,dz$$

(a) over the whole ellipsoid; and

(b) over that part of it in the first quadrant:

$$x \geq 0, \quad y \geq 0, \quad \text{and} \quad z \geq 0, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

13. Find the volume of the "ice cream cone" defined by the inequalities $x^2 + y^2 \leq \frac{1}{5}z^2$, and $0 \leq z \leq 5 + \sqrt{5 - x^2 - y^2}$.

14. Let ρ, θ, ϕ be spherical coordinates in $\mathbb{R}^3$ and suppose that a surface surrounding the origin is described by a continuous positive function $\rho = f(\theta, \phi)$. Show that the volume enclosed by the surface is

$$V = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} [f(\theta, \phi)]^3 \sin\phi\, d\phi\, d\theta.$$

15. Using an appropriate change of variables, evaluate

$$\iint_B \exp\,[(y - x)/(y + x)]\,dx\,dy,$$

where B is the interior of the triangle with vertices at $(0, 0)$, $(0, 1)$, and $(1, 0)$.

16. Suppose the density of a solid of radius R is given by $(1 + d^3)^{-1}$, where d is the distance to the center of the sphere. Find the total mass of the sphere.

17. The density of the material of a spherical shell whose inner radius is 1 m and whose outer radius is 2 m is $0.4d^2$ g/cm^3, where d is the distance to the center of the sphere in meters. Find the total mass of the shell.

18. If the shell in Exercise 17 were dropped into a large tank of pure water, would it float? What if the shell leaked? (Assume that the density of water is exactly 1 g/cm^3.)

19. The temperature at points in the cube $C = \{(x, y, z) \mid -1 \leq x \leq 1, -1 \leq y \leq 1, \text{and } -1 \leq z \leq 1\}$ is $32d^2$, where d is the distance to the origin.

(a) What is the average temperature?

(b) At what points of the cube is the temperature equal to the average temperature?

20. Use cylindrical coordinates to find the center of mass of the region defined by

$$y^2 + z^2 \leq \frac{1}{4}, \qquad (x - 1)^2 + y^2 + z^2 \leq 1, \qquad x \geq 1.$$

21. Find the center of mass of the solid hemisphere

$$V = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq a^2 \text{ and } z \geq 0\},$$

if the density is constant.

22. Evaluate $\iint_B e^{-x^2-y^2}\,dx\,dy$, where B consists of those (x, y) satisfying $x^2 + y^2 \leq 1$ and $y \leq 0$.

23. Evaluate

$$\iiint_S \frac{dx\,dy\,dz}{(x^2 + y^2 + z^2)^{3/2}},$$

where S is the solid bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $a > b > 0$.

24. Evaluate $\iiint_D (x^2 + y^2 + z^2)xyz\,dx\,dy\,dz$ over each of the following regions.

(a) The sphere $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq R^2\}$

(b) The hemisphere $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq R^2 \text{ and } z \geq 0\}$

(c) The octant $D = \{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0, \text{and } z^2 + y^2 + z^2 \leq R^2\}$

25. Let C be the cone-shaped region $\{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\}$ in $\mathbb{R}^3$ and evaluate the integral $\iiint_C (1 + \sqrt{x^2 + y^2})\,dx\,dy\,dz$.

26. Find $\iiint_{\mathbb{R}^3} f(x, y, z)\,dx\,dy\,dz$, where $f(x, y, z) = \exp\,[-(x^2 + y^2 + z^2)^{3/2}]$.

27. The *flexural rigidity EI* of a uniform beam is the product of its Young's modulus of elasticity E and the moment of inertia I of the cross section of the beam with respect to a horizontal line l passing through the center of gravity of this cross section. Here

$$I = \iint_R [d(x, y)]^2\,dx\,dy,$$

where $d(x, y) =$ the distance from (x, y) to l and $R =$ the cross section of the beam being considered.

(a) Assume that the cross section R is the rectangle $-1 \leq x \leq 1$, $-1 \leq y \leq 2$, and l is the line $y = 1/2$. Find I.

(b) Assume the cross section R is a circle of radius 4 and l is the x axis. Find I, using polar coordinates.

28. Find, $\iiint_{\mathbb{R}^3} f(x, y, z)\, dx\, dy\, dz$, where

$$f(x, y, z) = \frac{1}{[1 + (x^2 + y^2 + z^2)^{3/2}]^{3/2}}.$$

29. Suppose D is the unbounded region of $\mathbb{R}^2$ given by the set of (x, y) with $0 \leq x < \infty$, $0 \leq y \leq x$. Let $f(x, y) = x^{-3/2} e^{y-x}$. Does the improper integral $\iint_D f(x, y)\, dx\, dy$ exist?

30. If the world were two-dimensional, the laws of physics would predict that the gravitational potential of a mass point is proportional to the logarithm of the distance from the point. Using polar coordinates, write an integral giving the gravitational potential of a disk of constant density.

31. (a) Evaluate the improper integral

$$\int_0^{\infty} \int_0^{y} x e^{-y^3}\, dx\, dy.$$

(b) Evaluate

$$\iint_B (x^4 + 2x^2 y^2 + y^4)\, dx\, dy,$$

where B is the portion of the disk of radius 2 [centered at $(0, 0)$ in the first quadrant].

32. Let f be a nonnegative function on an x-simple or a y-simple region $D \subset \mathbb{R}^2$ and that is continuous except for points on the boundary of D and at most finitely many points interior to D. Give a suitable definition of $\iint_D f\, dA$.

33. Evaluate $\iint_{\mathbb{R}^2} f(x, y)\, dx\, dy$, where $f(x, y) = 1/(1 + x^2 + y^2)^{3/2}$. (HINT: You may assume that changing variables and Fubini's theorem are valid for improper integrals.)