solution First note that we cannot *easily* integrate this function using iterated integrals (try it!). Hence (employing the strategy in the quote that opened this chapter), let us try a change of variables. The transformation into spherical coordinates seems appropriate, because then the entire quantity $x^2+y^2+z^2$ can be replaced by one variable, namely, ρ^2. If W^* is the region such that

$$0 \le \rho \le 1, \qquad 0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi,$$

we can apply formula (10) and write

$$\iiint_W \exp{(x^2+y^2+z^2)^{3/2}}\, dV = \iiint_{W^*} \rho^2 e^{\rho^3} \sin\phi \, d\rho\, d\theta\, d\phi.$$

This integral equals the iterated integral

$$\begin{aligned}\int_0^1 \int_0^{\pi} \int_0^{2\pi} e^{\rho^3} \rho^2 \sin\phi \, d\theta\, d\phi\, d\rho &= 2\pi \int_0^1 \int_0^{\pi} e^{\rho^3} \rho^2 \sin\phi\, d\phi\, d\rho \\ &= -2\pi \int_0^1 \rho^2 e^{\rho^3} [\cos\phi]_0^{\pi}\, d\rho \\ &= 4\pi \int_0^1 e^{\rho^3} \rho^2\, d\rho = \frac{4}{3}\pi \int_0^1 e^{\rho^3}(3\rho^2)\, d\rho \\ &= \left[\frac{4}{3}\pi e^{\rho^3}\right]_0^1 = \frac{4}{3}\pi(e-1). \quad \blacktriangle\end{aligned}$$

example 7 Let W be the ball of radius R and center $(0,0,0)$ in $\mathbb{R}^3$. Find the volume of W.

solution The volume of W is $\iiint_W dx\,dy\,dz$. This integral may be evaluated by reducing it to iterated integrals or by regarding W as a volume of revolution, but let us evaluate it here by using spherical coordinates. We get

$$\begin{aligned}\iiint_W dx\,dy\,dz &= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{R^3}{3} \int_0^{\pi} \int_0^{2\pi} \sin\phi\, d\theta\, d\phi \\ &= \frac{2\pi R^3}{3} \int_0^{\pi} \sin\phi\, d\phi = \frac{2\pi R^3}{3}\{-[\cos(\pi) - \cos(0)]\} = \frac{4\pi R^3}{3},\end{aligned}$$

which is the standard formula for the volume of a solid sphere. ▲

exercises

1. Suggest a substitution/transformation that will simplify the following integrands, and find their Jacobians.

   (a) $\iint_R (3x+2y)\sin(x-y)\, dA$

   (b) $\iint_R e^{(-4x+7y)} \cos(7x-2y)\, dA$

2. Suggest a substitution/transformation that will simplify the following integrands, and find their Jacobians.

   (a) $\iint_R (5x+y)^3 (x+9y)^4\, dA$

   (b) $\iint_R x\sin(6x+7y) - 3y\sin(6x+7y)\, dA$

3. Let D be the unit disk: $x^2 + y^2 \leq 1$. Evaluate

$$\iint_D \exp(x^2 + y^2)\, dx\, dy$$

by making a change of variables to polar coordinates.

4. Let D be the region $0 \leq y \leq x$ and $0 \leq x \leq 1$. Evaluate

$$\iint_D (x + y)\, dx\, dy$$

by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly by using an iterated integral.

5. Let $T(u, v) = (x(u, v), y(u, v))$ be the mapping defined by $T(u, v) = (4u, 2u + 3v)$. Let D^* be the rectangle $[0, 1] \times [1, 2]$. Find $D = T(D^*)$ and evaluate

(a) $\iint_D xy\, dx\, dy$

(b) $\iint_D (x - y)\, dx\, dy$

by making a change of variables to evaluate them as integrals over D^*.

6. Repeat Exercise 5 for $T(u, v) = (u, v(1 + u))$.

7. Evaluate

$$\iint_D \frac{dx\, dy}{\sqrt{1 + x + 2y}},$$

where $D = [0, 1] \times [0, 1]$, by setting $T(u, v) = (u, v/2)$ and evaluating an integral over D^*, where $T(D^*) = D$.

8. Define $T(u, v) = (u^2 - v^2, 2uv)$. Let D^* be the set of (u, v) with $u^2 + v^2 \leq 1$, $u \geq 0$, $v \geq 0$. Find $T(D^*) = D$. Evaluate $\iint_D dx\, dy$.

9. Let $T(u, v)$ be as in Exercise 8. By making a change of variables, "formally" evaluate the "improper" integral

$$\iint_D \frac{dx\, dy}{\sqrt{x^2 + y^2}}.$$

[NOTE: This integral (and the one in the next exercise) is *improper*, because the integrand $1/\sqrt{x^2 + y^2}$ is neither continuous nor bounded on the domain of integration. (The theory of improper integrals is discussed in Section 6.4.)]

10. Calculate $\iint_R \frac{1}{x + y}\, dy\, dx$, where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$, $x + y = 4$, by using the mapping $T(u, v) = (u - uv, uv)$.

11. Evaluate $\iint_D (x^2 + y^2)^{3/2} dx\, dy$, where D is the disk $x^2 + y^2 \leq 4$.

12. Let D^* be a v-simple region in the uv plane bounded by $v = g(u)$ and $v = h(u) \leq g(u)$ for $a \leq u \leq b$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given by $x = u$ and $y = \psi(u, v)$, where ψ is of class C^1 and $\partial\psi/\partial v$ is never zero. Assume that $T(D^*) = D$ is a y-simple region; show that if $f\colon D \to \mathbb{R}$ is continuous, then

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(u, \psi(u, v)) \left| \frac{\partial \psi}{\partial v} \right| du\, dv.$$

13. Use double integrals to find the area inside the curve $r = 1 + \sin\theta$.

14. (a) Express $\int_0^1 \int_0^{x^2} xy\, dy\, dx$ as an integral over the triangle D^*, which is the set of (u, v) where $0 \leq u \leq 1$, $0 \leq v \leq u$. (HINT: Find a one-to-one mapping T of D^* onto the given region of integration.)

(b) Evaluate this integral directly and as an integral over D^*.

15. Integrate $ze^{x^2+y^2}$ over the cylinder $x^2 + y^2 \leq 4$, $2 \leq z \leq 3$.

16. Let D be the unit disk. Express $\iint_D (1 + x^2 + y^2)^{3/2} dx\, dy$ as an integral over $[0, 1] \times [0, 2\pi]$ and evaluate.

17. Using polar coordinates, find the area bounded by the *lemniscate* $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$.

18. Redo Exercise 15 of Section 5.3 using a change of variables and compare the effort involved in each method.

19. Calculate $\iint_R (x + y)^2 e^{x-y} dx\, dy$, where R is the region bounded by $x + y = 1$, $x + y = 4$, $x - y = -1$, and $x - y = 1$.

20. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(u, v, w) = (u \cos v \cos w, u \sin v \cos w, u \sin w).$$

(a) Show that T is onto the unit sphere; that is, every (x, y, z) with $x^2 + y^2 + z^2 = 1$ can be written as $(x, y, z) = T(u, v, w)$ for some (u, v, w).

(b) Show that T is not one-to-one.

21. Integrate $x^2 + y^2 + z^2$ over the cylinder $x^2 + y^2 \leq 2$, $-2 \leq z \leq 3$.

22. Evaluate $\int_0^{\infty} e^{-4x^2}\, dx$.

23. Let B be the unit ball. Evaluate

$$\iiint_B \frac{dx\, dy\, dz}{\sqrt{2 + x^2 + y^2 + z^2}}$$

by making the appropriate change of variables.

24. Evaluate $\iint_A [1/(x^2 + y^2)^2]\, dx\, dy$, where A is determined by the conditions $x^2 + y^2 \leq 1$ and $x + y \geq 1$.

25. Evaluate $\iiint_W \frac{dx\, dy\, dz}{(x^2 + y^2 + z^2)^{3/2}}$, where W is the solid bounded by the two spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $0 < b < a$.

26. Use spherical coordinates to evaluate:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} \frac{\sqrt{x^2 + y^2 + z^2}}{1 + [x^2 + y^2 + z^2]^2}\, dz\, dy\, dx$$

27. Let D be a triangle in the (x, y) plane with vertices $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(1, 0)$. Evaluate:

$$\iint_D \cos \pi \left(\frac{x - y}{x + y}\right) dx\, dy$$

by making the appropriate change of variables.

28. Evaluate $\iint_D x^2 dx\, dy$, where D is determined by the two conditions $0 \leq x \leq y$ and $x^2 + y^2 \leq 1$.

29. Integrate $\sqrt{x^2 + y^2 + z^2}\, e^{-(x^2+y^2+z^2)}$ over the region described in Exercise 25.

30. Evaluate the following by using cylindrical coordinates.

(a) $\iiint_B z\, dx\, dy\, dz$, where B is the region within the cylinder $x^2 + y^2 = 1$ above the xy plane and below the cone $z = (x^2 + y^2)^{1/2}$

(b) $\iiint_W (x^2 + y^2 + z^2)^{-1/2} dx\, dy\, dz$, where W is the region determined by the conditions $\frac{1}{2} \leq z \leq 1$ and $x^2 + y^2 + z^2 \leq 1$

31. Evaluate $\iint_B (x + y)\, dx\, dy$, where B is the rectangle in the xy plane with vertices at $(0, 1)$, $(1, 0)$, $(3, 4)$, and $(4, 3)$.

32. Evaluate $\iint_D (x + y)\, dx\, dy$, where D is the square with vertices at $(0, 0)$, $(1, 2)$, $(3, 1)$, and $(2, -1)$.

33. Let E be the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1$ where a, b, and c are positive.

(a) Find the volume of E.

(b) Evaluate

$$\iiint_E [(x^2/a^2) + (y^2/b^2) + (z^2/c^2)]\, dx\, dy\, dz.$$

(HINT: Change variables and then use spherical coordinates.)

34. Using spherical coordinates, compute the integral of $f(\rho, \phi, \theta) = 1/\rho$ over the region in the first octant of $\mathbb{R}^3$, which is bounded by the cones $\phi = \pi/4$, $\phi = \arctan 2$ and the sphere $\rho = \sqrt{6}$.

35. The mapping $T(u, v) = (u^2 - v^2, 2uv)$ transforms the rectangle $1 \leq u \leq 2$, $1 \leq v \leq 3$ of the uv plane into a region R of the xy plane.

(a) Show that T is one-to-one.

(b) Find the area of R using the change of variables formula.

36. Let R denote the region inside $x^2 + y^2 = 1$, but outsid $x^2 + y^2 = 2y$ with $x \geq 0$, $y \geq 0$.

(a) Sketch this region.

(b) Let $u = x^2 + y^2$, $v = x^2 + y^2 - 2y$. Sketch the region D in the uv plane, which corresponds to R under this change of coordinates.

(c) Compute $\iint_R xe^y dx\, dy$ using this change of coordinates.

37. Let D be the region bounded by $x^{3/2} + y^{3/2} = a^{3/2}$, fo $x \geq 0$, $y \geq 0$, and the coordinate axes $x = 0$, $y = 0$. Express $\iint_D f(x, y)\, dx\, dy$ as an integral over the triangle D^*, which is the set of points $0 \leq u \leq a$, $0 \leq v \leq a - u$. (Do not attempt to evaluate

38. Show that $S(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$, the spherical change-of-coordinate mapping, one-to-one except on a set that is a union of finitely many graphs of continuous functions.