example 6 Evaluate

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dz\, dy\, dx.$$

Sketch the region W of integration and interpret.

solution

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dz\, dy\, dx = \int_0^1 \int_0^x (2 - x^2 - y^2)\, dy\, dx$$

$$= \int_0^1 \left(2x - x^3 - \frac{x^3}{3}\right) dx = 1 - \frac{1}{4} - \frac{1}{12} = \frac{2}{3}.$$

This integral is the volume of the region sketched in Figure 5.5.8.

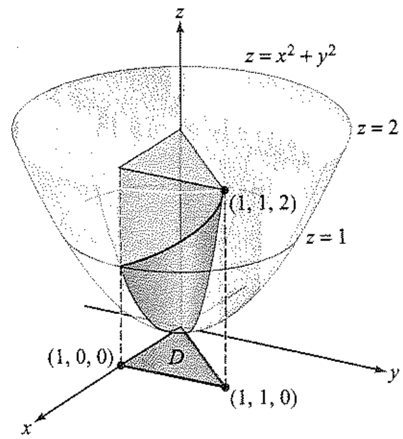


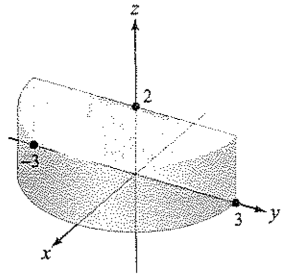
figure **5.5.8** The region W lies between the paraboloid $z = x^2 + y^2$ and the plane $z = 2$, and above the region D.

exercises

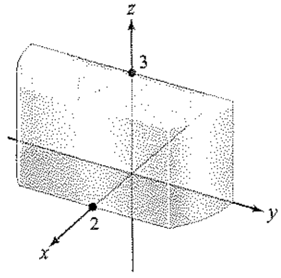
1. In parts (a) through (d) below, each iterated integral is an integral over a region D. Match the integral with the correct region of integration.

(a) $\displaystyle\int_0^2 \int_0^3 \int_{-\sqrt{9-x}}^{\sqrt{9-x}} dy\, dz\, dx$ (c) $\displaystyle\int_0^1 \int_0^x \int_0^y dz\, dy\, dx$

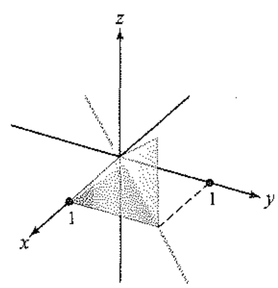
(b) $\displaystyle\int_0^2 \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\, dx\, dz$ (d) $\displaystyle\int_0^1 \int_0^y \int_0^x dz\, dx\, dy$



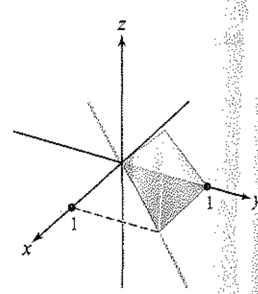
(i)



(ii)



(iii)



(iv)

2. Evaluate the following triple integral:

$$\iiint_W \sin x\, dx\, dy\, dz,$$

where W is the solid given by $0 \le x \le \pi$, $0 \le y \le 1$, and $0 \le z \le x$.

In Exercises 3 to 6, perform the indicated integration over the given box.

3. $\iiint_B x^2\,dx\,dy\,dz$, $B = [0, 1] \times [0, 1] \times [0, 1]$

4. $\iiint_B e^{-xy}\,y\,dx\,dy\,dz$, $B = [0, 1] \times [0, 1] \times [0, 1]$

5. $\iiint_B (2x+3y+z)\,dx\,dy\,dz$, $B = [0, 2]\times[-1, 1]\times[0, 1]$

6. $\iiint_B z e^{x+y}\,dx\,dy\,dz$, $B = [0, 1] \times [0, 1] \times [0, 1]$

In Exercises 7 to 10, describe the given region as an elementary region.

7. The region between the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$

8. The region cut out of the ball $x^2 + y^2 + z^2 \le 4$ by the elliptic cylinder $2x^2 + z^2 = 1$; that is, the region inside the cylinder and the ball

9. The region inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$

10. The region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$, and $x = z - y - 1$

Find the volume of the region in Exercises 11 to 14.

11. The region bounded by $z = x^2 + y^2$ and $z = 10 - x^2 - 2y^2$

12. The solid bounded by $x^2 + 2y^2 = 2$, $z = 0$, and $x + y + 2z = 2$

13. The solid bounded by $x = y$, $z = 0$, $y = 0$, $x = 1$, and $x + y + z = 0$

14. The region common to the intersecting cylinders $x^2 + y^2 \le a^2$ and $x^2 + z^2 \le a^2$

Evaluate the integrals in Exercises 15 to 23.

15. $\int_0^1 \int_1^2 \int_2^3 \cos[\pi(x + y + z)]\,dx\,dy\,dz$

16. $\int_0^1 \int_0^x \int_0^y (y + xz)\,dz\,dy\,dx$

17. $\iiint_W (x^2 + y^2 + z^2)\,dx\,dy\,dz$; W is the region bounded by $x + y + z = a$ (where $a > 0$), $x = 0$, $y = 0$, and $z = 0$.

18. $\iiint_W z\,dx\,dy\,dz$; W is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = 1$, with $x \ge 0$, $y \ge 0$.

19. $\iiint_W x^2 \cos z\,dx\,dy\,dz$; W is the region bounded by $z = 0$, $z = \pi$, $y = 0$, $y = 1$, $x = 0$, and $x + y = 1$.

20. $\int_0^2 \int_0^x \int_0^{x+y} dz\,dy\,dx$

21. $\iiint_W (1 - z^2)\,dx\,dy\,dz$; W is the pyramid with top vertex at $(0, 0, 1)$ and base vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$.

22. $\iiint_W (x^2 + y^2)\,dx\,dy\,dz$; W is the same pyramid as in Exercise 21.

23. $\int_0^1 \int_0^{2x} \int_{x^2+y^2}^{x+y} dz\,dy\,dx.$

24. (a) Sketch the region for the integral
$$\int_0^1 \int_0^x \int_0^y f(x, y, z)\,dz\,dy\,dx.$$
(b) Write the integral with the integration order $dx\,dy\,dz$.

For the regions in Exercises 25 to 28, find the appropriate limits $\phi_1(x)$, $\phi_2(x)$, $\gamma_1(x, y)$, and $\gamma_2(x, y)$, and write the triple integral over the region W as an iterated integral in the form

$$\iiint_W f\,dV = \int_a^b \left\{ \int_{\phi_1(x)}^{\phi_2(x)} \left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\,dz \right] dy \right\} dx.$$

25. $W = \{(x, y, z) \mid \sqrt{x^2 + y^2} \leq z \leq 1\}$

26. $W = \{(x, y, z) \mid \frac{1}{2} \leq z \leq 1 \text{ and } x^2 + y^2 + z^2 \leq 1\}$

27. $W = \{(x, y, z) \mid x^2 + y^2 \leq 1, z \geq 0$ and $x^2 + y^2 + z^2 \leq 4\}$

28. $W = \{(x, y, z) \mid |x| \leq 1, |y| \leq 1, z \geq 0$ and $x^2 + y^2 + z^2 \leq 1\}$

29. Show that the formula using triple integrals for the volume under the graph of a positive function $f(x, y)$, on an elementary region D in the plane, reduces to the double integral of f over D.

30. Let W be the region bounded by the planes $x = 0, y = 0, z = 0, x + y = 1$, and $z = x + y$.
   (a) Find the volume of W.
   (b) Evalute $\iiint_W x\,dx\,dy\,dz$.
   (c) Evalute $\iiint_W y\,dx\,dy\,dz$.

31. Let f be continuous and let B_ε be the ball of radius ε centered at the point (x_0, y_0, z_0). Let vol (B_ε) be the volume of B_ε. Prove that

$$\lim_{\varepsilon \to 0} \frac{1}{\text{vol}\,(B_\varepsilon)} \iiint_{B_\varepsilon} f(x, y, z)\,dV = f(x_0, y_0, z_0).$$

review exercises for chapter 5

Evaluate the integrals in Exercises 1 to 4.

1. $\displaystyle\int_0^3 \int_{-x^2+1}^{x^2+1} xy\,dy\,dx$

2. $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 (x + y)^2 dy\,dx$

3. $\displaystyle\int_0^1 \int_{e^x}^{e^{2x}} x \ln y\,dy\,dx$

4. $\displaystyle\int_0^1 \int_1^2 \int_2^3 \cos\,[\pi(x + y + z)]\,dx\,dy\,dz.$

Reverse the order of integration of the integrals in Exercises 5 to 8 and evaluate.

5. The integral in Exercise 1

6. The integral in Exercise 2

7. The integral in Exercise 3

8. The integral in Exercise 4

9. Evaluate the integral $\int_0^1 \int_0^x \int_0^y (y + xz)\,dz\,dy\,dx$.

10. Evaluate $\int_0^1 \int_y^{y^2} e^{x/y}\,dx\,dy$.

11. Evaluate $\int_0^1 \int_0^{(\arcsin y)/y} y \cos xy\,dx\,dy$.

12. Change the order of integration and evaluate

$$\int_0^2 \int_{y/2}^1 (x + y)^2 dx\,dy.$$

13. Show that evaluating $\iint_D dx\,dy$, where D is a y-simple region, reproduces the formula from one-variable calculus for the area between two curves.

14. Change the order of integration and evaluate

$$\int_0^1 \int_{y^{1/2}}^1 (x^2 + y^3 x)\,dx\,dy.$$

15. Let D be the region in the xy plane inside the unit circle $x^2 + y^2 = 1$. Evaluate $\iint_D f(x, y)\,dx\,dy$ in each of the following cases:
   (a) (a)] $f(x, y) = xy$
   (b) $f(x, y) = x^2 y^2$
   (c) $f(x, y) = x^3 y^3$

16. Find $\iint_D y[1 - \cos(\pi x/4)]dx\,dy$, where D is the region in Figure 5.R.1.

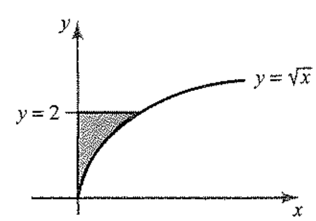


figure **5.R.1** The region of integration for Exercise 16.

Evaluate the integrals in Exercises 17 to 24. Sketch and identify the type of the region (corresponding to the way the integral is written).

17. $\displaystyle\int_0^{\pi}\int_{\sin x}^{3\sin x} x(1+y)\,dy\,dx$

18. $\displaystyle\int_0^{1}\int_{x-1}^{x\cos(\pi x/2)} (x^2+xy+1)\,dy\,dx$

19. $\displaystyle\int_{-1}^{1}\int_{y^{2/3}}^{(2-y)^2} \left(\frac{3}{2}\sqrt{x}-2y\right)dx\,dy$

20. $\displaystyle\int_0^{2}\int_{-3(\sqrt{4-x^2})/2}^{3(\sqrt{4-x^2})/2} \left(\frac{5}{\sqrt{2+x}}+y^3\right)dy\,dx$

21. $\displaystyle\int_0^{1}\int_0^{x^2} (x^2+xy-y^2)\,dy\,dx$

22. $\displaystyle\int_2^{4}\int_{y^2-1}^{y^3} 3\,dx\,dy$

23. $\displaystyle\int_0^{1}\int_{x^2}^{x} (x+y)^2\,dy\,dx$

24. $\displaystyle\int_0^{1}\int_0^{3y} e^{x+y}\,dx\,dy$

In Exercises 25 to 27, integrate the given function f over the given region D.

25. $f(x,y)=x-y$; D is the triangle with vertices $(0,0)$, $(1,0)$, and $(2,1)$.

26. $f(x,y)=x^3y+\cos x$; D is the triangle defined by $0\le x\le \pi/2$, $0\le y\le x$.

27. $f(x,y)=x^2+2xy^2+2$; D is the region bounded by the graph of $y=-x^2+x$, the x axis, and the lines $x=0$ and $x=2$.

In Exercises 28 and 29, sketch the region of integration, interchange the order, and evaluate.

28. $\displaystyle\int_1^{4}\int_1^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

29. $\displaystyle\int_0^{1}\int_{1-y}^{1} (x+y^2)\,dx\,dy$

30. Show that

$$4e^5 \le \iint_{[1,3]\times[2,4]} e^{x^2+y^2}\,dA \le 4e^{25}.$$

31. Show that

$$4\pi \le \iint_D (x^2+y^2+1)\,dx\,dy \le 20\pi,$$

where D is the disk of radius 2 centered at the origin.

32. Suppose W is a ***path-connected region***; that is, given any two points of W there is a continuous path joining them. If f is a continuous function on W, use the intermediate-value theorem to show that there is at least one point in W at which the value of f is equal to the average of f over W; that is, the integral of f over W divided by the volume of W. (Compare this with the mean-value theorem for double integrals.) What happens if W is not connected?

33. Prove: $\int_0^x [\int_0^t F(u)\,du]\,dt = \int_0^x (x-u)F(u)\,du$.

Evaluate the integrals in Exercises 34 to 36.

34. $\displaystyle\int_0^{1}\int_0^{z}\int_0^{y} xy^2z^3\,dx\,dy\,dz$

35. $\displaystyle\int_0^{1}\int_0^{y}\int_0^{x/\sqrt{3}} \frac{x}{x^2+z^2}\,dz\,dx\,dy$

36. $\displaystyle\int_1^{2}\int_1^{z}\int_{1/y}^{2} yz^2\,dx\,dy\,dz$

37. Write the iterated integral $\int_0^1\int_{1-x}^1\int_x^1 f(x,y,z)\,dz\,dy\,dx$ as an integral over a region in $\mathbb{R}^3$ and then rewrite it in five other possible orders of integration.