example 4 Compute the double integral $\iint_S \cos x \sin y \, dx\, dy$, where S is the square $[0, \pi/2] \times [0, \pi/2]$ (see Figure 5.1.9).

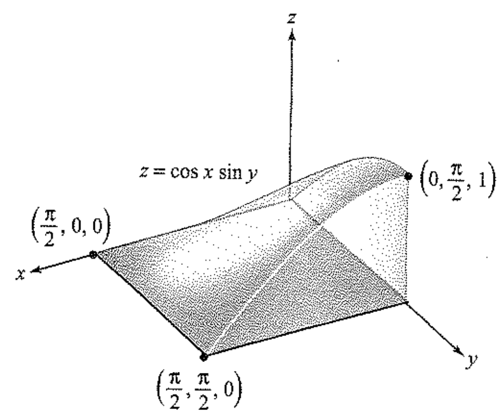

figure **5.1.9** Volume under $z = \cos x \sin y$ and over the rectangle $[0, \pi/2] \times [0, \pi/2]$.

solution By equation (2),

$$\iint_S \cos x \sin y \, dx\, dy = \int_0^{\pi/2} \left[\int_0^{\pi/2} \cos x \sin y \, dx \right] dy$$
$$= \int_0^{\pi/2} \sin y \left[\int_0^{\pi/2} \cos x \, dx \right] dy = \int_0^{\pi/2} \sin y \, dy = 1. \quad \blacktriangle$$

In the next section, we shall use Riemann sums to rigorously define the double integral for a large class of functions of two variables without recourse to the notion of volume. Although we shall drop the requirement that $f(x, y) \geq 0$, equations (1) and (2) will remain valid. Therefore, the iterated integral will again provide the key to computing the double integral. In Section 5.3, we treat double integrals over regions more general than rectangles.

Finally, we remark that it is common to delete the brackets in iterated integrals such as equations (1) and (2) and to write

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \quad \text{in place of} \quad \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

and

$$\int_c^d \int_a^b f(x, y)\, dx\, dy \quad \text{in place of} \quad \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy.$$

exercises

1. Evaluate the following iterated integrals:

(a) $\int_0^1 \int_0^1 (1 - x^3 + xy)\, dx\, dy$

(b) $\int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \cos x \sin y \, dx\, dy$

(c) $\int_1^2 \int_2^4 \left(\frac{x}{y} + \frac{y}{x} \right) dx\, dy$

(d) $\int_0^{\pi/4} \int_0^{\pi/4} \tan x \sec^2 y \, dx\, dy$

2. Evaluate the integrals in Exercise 1 by integrating first with respect to y and then with respect to x.

3. Evaluate the following iterated integrals:

(a) $\int_{-1}^{1}\int_{0}^{1}(x^4 y + y^2)\,dy\,dx$

(b) $\int_{0}^{\pi/2}\int_{0}^{1}(y\cos x + 2)\,dy\,dx$

(c) $\int_{0}^{1}\int_{0}^{1}(xye^{x+y})\,dy\,dx$

(d) $\int_{-1}^{0}\int_{1}^{2}(-x\log y)\,dy\,dx$

4. Evaluate the integrals in Exercise 3 by integrating with respect to x and then with respect to y. [The solution to part (b) only is in the Study Guide to this text.]

5. Use Cavalieri's principle to show that the volumes of two cylinders with the same base and height are equal (see Figure 5.1.10).

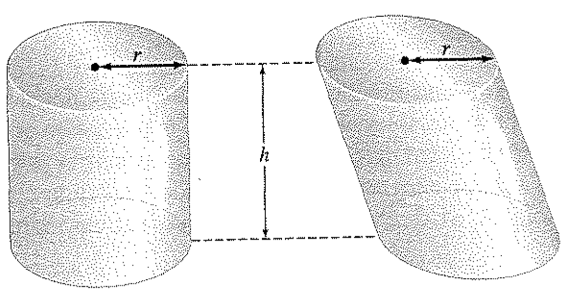

figure **5.1.10** Two cylinders with the same base and height have the same volume.

6. Using Cavalieri's principle, compute the volume of the structure shown in Figure 5.1.11; each cross section is a rectangle of length 5 and width 3.

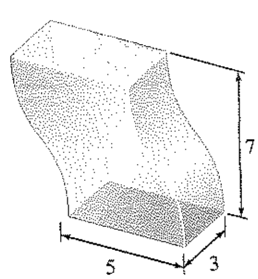

figure **5.1.11** Compute this volume.

7. A lumberjack cuts out a wedge-shaped piece W of a cylindrical tree of radius r obtained by making two saw cuts to the tree's center, one horizontally and one at an angle θ. Compute the volume of the wedge W using Cavalieri's principle. (See Figure 5.1.12.)

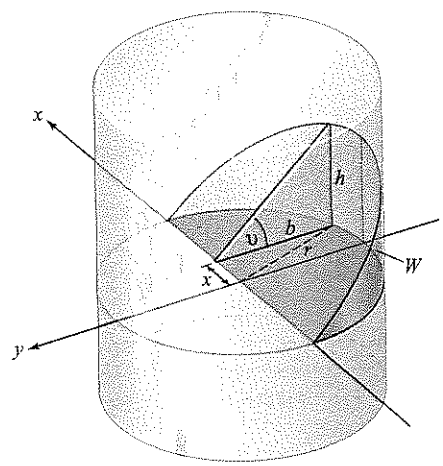

figure **5.1.12** Find the volume of W.

8. (a) Show that the volume of the solid of revolution shown in Figure 5.1.13(a) is

$$\pi\int_{a}^{b}[f(x)]^2\,dx.$$

(b) Show that the volume of the region obtained by rotating the region under the graph of the parabola $y = -x^2 + 2x + 3$, $-1 \le x \le 3$, about the x axis is $512\pi/15$ [see Figure 5.1.13(b)].

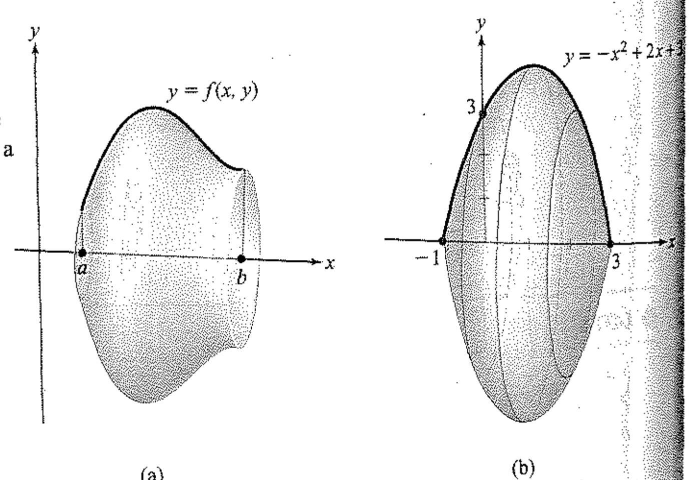

figure **5.1.13** The solid of revolution (a) has volume $\pi\int_a^b (f(x))^2\,dx$. Part (b) shows the region between the graph of $y = -x^2 + 2x + 3$ and the x axis rotated about the x axis.

Evaluate the double integrals in Exercises 9 to 11, where R is the rectangle $[0, 2] \times [-1, 0]$.

9. $\displaystyle\iint_R (x^2y^2 + x)\, dy\, dx$

10. $\displaystyle\iint_R \left(|y| \cos \frac{1}{4}\pi x\right) dy\, dx$

11. $\displaystyle\iint_R \left(-xe^x \sin \frac{1}{2}\pi y\right) dy\, dx$

12. Evaluate the iterated integral:

$$\int_1^3 \int_1^2 \frac{xy}{(x^2 + y^2)^{3/2}}\, dx\, dy.$$

13. Evaluate the iterated integral:

$$\int_0^1 \int_0^1 (3x + 2y)^7\, dx\, dy.$$

14. Find the volume bounded by the graph of $f(x, y) = 1 + 2x + 3y$, the rectangle $[1, 2] \times [0, 1]$, and the four vertical sides of the rectangle R, as in Figure 5.1.1.

15. Repeat Exercise 14 for the function $f(x, y) = x^4 + y^2$ and the rectangle $[-1, 1] \times [-3, -2]$.

5.2 The Double Integral Over a Rectangle

We are ready to give a rigorous definition of the double integral as the limit of a sequence of sums. This will then be used to *define* the volume of the region under the graph of a function $f(x, y)$. We shall not require that $f(x, y) \geq 0$; but if $f(x, y)$ assumes negative values, we shall interpret the integral as a signed volume, just as for the area under the graph of a function of one variable. In addition, we shall discuss some of the fundamental algebraic properties of the double integral and prove Fubini's theorem, which states that the double integral can be calculated as an iterated integral. To begin, let us establish some notation for partitions and sums.

Definition of the Integral

Consider a closed rectangle $R \subset \mathbb{R}^2$; that is, R is a Cartesian product of two intervals: $R = [a, b] \times [c, d]$. By a ***regular partition*** of R of order n we mean the two ordered collections of $n + 1$ equally spaced points $\{x_j\}_{j=0}^n$ and $\{y_k\}_{k=0}^n$; that is, the points satisfying

$$a = x_0 < x_1 < \cdots < x_n = b, \qquad c = y_0 < y_1 < \cdots < y_n = d$$

and

$$x_{j+1} - x_j = \frac{b-a}{n}, \qquad y_{k+1} - y_k = \frac{d-c}{n}$$

(see Figure 5.2.1).

A function $f(x, y)$ is said to be ***bounded*** if there is a number $M > 0$ such that $-M \leq f(x, y) \leq M$ for all (x, y) in the domain of f. A continuous function on a *closed* rectangle is always bounded, but, for example, $f(x, y) = 1/x$ on $(0, 1] \times [0, 1]$ is continuous but is not bounded, because $1/x$ becomes arbitrarily large for x near 0. The rectangle $(0, 1] \times [0, 1]$ is not closed, because the endpoint 0 is missing in the first factor.