1. Prove that a regular 7-gon is not constructible.

**Answer:** A regular 7-gon is constructible if and only if \( \zeta = e^{\frac{2\pi i}{7}} \) is constructible. It is clear that \( \zeta \) is a root of \( x^7 - 1 \). It is also clear that \( x^7 - 1 = (x - 1)g(x) \), where

\[
g(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1.
\]

We know that \( \zeta \) is not a root of \( x - 1 \); so, \( \zeta \) is a root of \( g(x) \). We also proved that \( g(x) \) is irreducible because 6 is equal to \( p - 1 \), where \( p \) is the prime integer 7. (Our proof consisted of viewing \( g(x) = f(x - 1) \) for some polynomial \( f(y) \). We used the Eisentstein criteria with \( p = 7 \) to show that \( f(y) \) is irreducible; hence, \( g(x) \) is also irreducible.) At any rate, \( g(x) \) is the minimal polynomial of \( \zeta \). It follows that \( \text{dim}_\mathbb{Q}\mathbb{Q}[\zeta] = 6 \). We proved that if \( u \) is a constructible number then \( \text{dim}_\mathbb{Q}\mathbb{Q}[u] = 2^n \), for some integer \( n \). We know that 6 is not a power of 2 and therefore we conclude that \( \zeta \) is not constructible.