Review sheet for Exam 3 – First Installment

Be able to do all of the assigned Homework problems, all of the problems on Exams 1 and 2, and all of the problems on the Review sheets for Exams 1 and 2.

1. Be able to define: prime ideal, maximal ideal, Principal Ideal Domain, irreducible element, Unique Factorization domain.

2. Let \( f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \) be a polynomial in \( \mathbb{Z}[x] \). Suppose that \( a \) and \( b \) are relatively prime integers with \( f\left(\frac{a}{b}\right) = 0 \). Prove that \( a \) divides \( c_0 \) in \( \mathbb{Z} \) and \( b \) divides \( c_n \) in \( \mathbb{Z} \).

3. Write \( \frac{1}{\sqrt[15]{2}} \), \( \frac{1}{1 + \sqrt[15]{2}} \), and \( \frac{1}{1 + 2 \sqrt[15]{2} + 3 (\sqrt[15]{2})^2} \) in the form \( a + b \sqrt[15]{2} + c (\sqrt[15]{2})^2 \) with \( a \), \( b \), \( c \) in \( \mathbb{Q} \).

4. Let \( f \) be a polynomial in \( \mathbb{Z}[x] \). Suppose that the coefficients of \( f \) are relatively prime. Prove that \( f \) is irreducible in \( \mathbb{Z}[x] \) if and only if \( f \) is irreducible in \( \mathbb{Q}[x] \).

5. Let \( R \) be a domain. Suppose that there exists a field \( F \) with \( F \subseteq R \) and \( \dim_F R < \infty \). Prove that \( R \) is a field.

6. Give an example of a ring \( R \) and a field \( F \) with \( F \subseteq R \), \( \dim_F R < \infty \), and \( R \) is not a field.

7. Let \( R \) be a domain in which every ideal is finitely generated. Let \( r \) be an element of \( R \) which is not zero and not a unit. Prove that \( r \) is equal to a finite product of irreducible elements.