Math 546, Exam 1, Summer 2002

PRINT Your Name: ___________________________

There are 8 problems on 5 pages. Problems 1 and 2 are worth 7 points each. Each of the other problems is worth 6 points.

1. Define “group”. Use complete sentences.

2. Define “subgroup”. Use complete sentences.

3. Define \( \star \) on \( \mathbb{Q} \setminus \{0\} \) by \( a \star b = \frac{a}{b} \). Is \( (\mathbb{Q} \setminus \{0\}, \star) \) a group? Why or why not?

4. Recall that \( \text{GL}_2(\mathbb{R}) \) represents the group of invertible \( 2 \times 2 \) matrices with real number entries. The operation in \( \text{GL}_2(\mathbb{R}) \) is matrix multiplication. The matrix

\[
A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}
\]

is an element of \( \text{GL}_2(\mathbb{R}) \). What is \( A \)'s inverse?

5. Let \( T = \mathbb{R} \setminus \{-2\} \). Define \( \star \) on \( T \) by \( a \star b = ab + 2a + 2b + 2 \). Proof that \( (T, \star) \) is a group.

6. Recall that \( D_3 \) is the smallest subgroup of the group of rigid motions which contains \( \rho \) and \( \sigma \), where \( \rho \) is rotation counter clockwise by 120° fixing the origin and \( \sigma \) is reflection of the \( xy \) plane across the \( x \) axis. List 4 subgroups of \( D_3 \) in addition to \( D_3 \) and \{id\}. (I do not need to see any details.)

7. The Dihedral group \( D_4 \) consists of 8 elements \( \text{id} \), \( \rho \), \( \rho^2 \), \( \rho^3 \), \( \sigma \), \( \sigma \rho \), \( \sigma \rho^2 \), and \( \sigma \rho^3 \). In class we calculated that \( \rho \sigma = \sigma \rho^3 \), \( \rho^4 = \text{id} \), and \( \sigma^2 = \text{id} \). Express \( \rho^2 \sigma \rho \sigma \) in the form \( \sigma^i \rho^j \) for some integers \( i \) and \( j \), with \( 0 \leq i \leq 1 \), and \( 0 \leq j \leq 3 \).

8. Consider \( L = \{ n \in \mathbb{Z} \mid n \leq 7 \} \). For \( a \) and \( b \) in \( L \), define \( a \star b = \min\{a, b\} \). Does \( (L, \star) \) have an identity element? If yes, what is it and why does it work? If no, why not? (I know that \( (L, \star) \) is not a group. You do not have to show that, but you do have to answer my question.)