Let $G$ be a finite group with an even number of elements. Prove that there must exist an element $a \in G$ with $a \neq \text{id}$, but $a^2 = \text{id}$.

**ANSWER:** Observe that $G$ is the disjoint union of the sets

$$Y = \{g \in G \mid g^2 = \text{id}\} \quad \text{and} \quad N = \{g \in G \mid g^2 \neq \text{id}\}.$$ 

The set $Y$ always contains at least one element, namely $\text{id}$. Observe that if $g \in N$, then $g^{-1}$ is also in $N$ and $g \neq g^{-1}$. It follows that $N$ may be partitioned into a collection of subsets each of which consists of a pair of elements which are inverses of one another. Thus, $N$ contains an even number of elements. The hypothesis ensures that the group $G$ contains an even number of elements. We conclude that $Y$ contains an even number of elements. Since $Y$ contains at least one element, we now know that $Y$ must contain at least two elements. In other words, there does exist an element $g$ in $G$ with $g \neq \text{id}$, but $g^2 = \text{id}$. 