Quiz for March 4, 2004

Let $G$ be a group with $a$ and $b$ in $G$. Assume that $o(a)$ and $o(b)$ are finite and relatively prime, and that $ab = ba$. Prove that $o(ab) = o(a)o(b)$.

**ANSWER:** Let $\ell = o(a)$, $m = o(b)$, and $n = o(ab)$. Since $\ell$, $m$ and $n$ all are positive integers, it suffices to prove that $n|\ell m$ and $\ell m|n$.

$n|\ell m$: The elements $a$ and $b$ commute; hence,

$$(ab)^{\ell m} = a^{\ell m}b^{\ell m} = (a^{\ell})^m(b^m)^{\ell} = \text{id}.$$ 

So, $(ab)^{\ell m}$ is the identity. It follows that $n$, which is the order of $ab$, must divide $\ell m$.

$\ell m|n$: Observe that

$$\text{id} = ((ab)^n)^{\ell} = (a^{\ell})^n(b^n)^{\ell} = b^n\ell.$$ 

The order of $b$ is $m$; thus, $m|n\ell$. The integers $m$ and $\ell$ are relatively prime; thus, $m|n$.

In a similar manner, we see that

$$\text{id} = ((ab)^n)^m = a^m(b^n)^m = a^{mn}.$$ 

The order of $a$ is $\ell$; thus, $\ell|mn$. The integers $\ell$ and $m$ are relatively prime; so, $\ell|n$.

Finally, we notice that $m|n$ and $\ell|n$, with $\ell$ and $m$ relatively prime. It follows that $m\ell|n$, and the proof is complete.