Quiz for February 5, 2004

Let $G$ be a group. Prove that the center of $G$ is a subgroup of $G$. (You probably have to tell me what the center of $G$ is.)

**ANSWER:**

The center of the group $G$ is the set

$$Z = \{ x \in G \mid xg = gx \text{ for all } g \in G \}.$$  

(In other words, the center of $G$ is the set which consists of all elements of $G$ which commute with every element of $G$.)

The set $Z$ is closed. Suppose $x$ and $y$ are in $Z$, we must show that $xy$ is in $Z$. Let $g$ be an arbitrary element of $G$. We must show that $xy$ commutes with $g$. Well, $xyg = xgy$ because $y \in Z$ and $xgy = gxy$ because $x \in Z$. Thus, $(xy)g = g(xy)$, and $xy \in Z$.

The set $Z$ is non-empty because the identity element of $G$ is in $Z$.

The inverse axiom is satisfied. Let $x$ be an element of $Z$. We know that $x$ has an inverse, called $x^{-1}$, in $G$. We must show that $x^{-1}$ is in $Z$. We must show that $x^{-1}$ commutes with every element of $G$. Let $g$ be an arbitrary element of $G$. We know that $xg = gx$ (because $x \in Z$). Multiply both sides of this equation on the left by $x^{-1}$ to get $g = x^{-1}gx$. Multiply both sides of this equation on the right by $x^{-1}$ to get $gx^{-1} = x^{-1}g$. We conclude that $x^{-1} \in Z$.

We proved the following result in class.

**Proposition.** Let $H$ be a non-empty subset of the group $(G, \ast)$. Suppose $H$ is closed under $\ast$. Suppose, also, that whenever $h \in H$, then the inverse of $h$ in $G$ is also an element of $H$. Then $H$ is a subgroup of $G$.

Apply the Proposition to conclude that $Z$ is a subgroup of $G$. 