section 2.3, page 75 1a, 3 (for $\sigma \tau$), 4, 5a, 6 (list all of the elements of $S_4$, use cycle notation), 7.


1. Let $G$ be a group. For each element $a$ in $G$, let $\lambda_a$ be the function from $G$ to $G$, which is defined by $\lambda_a(g) = ag$.

(a) Prove that $\lambda_a : G \rightarrow G$ is one-to-one and onto. (Once you have completed this part of the problem, then you have shown that each $\lambda_a$ is an element of $\text{Sym}(G)$.)

(b) Prove that the function $\lambda : G \rightarrow \text{Sym}(G)$, which is given by $\lambda(a) = \lambda_a$, is a group homomorphism. (If $a$ and $b$ are elements of $G$ then you must show that the FUNCTIONS $\lambda(ab)$ and $\lambda_a \circ \lambda_b$ are equal. One usually shows that two functions are equal by showing that they do the same thing to each element of the domain.)

(c) Prove that $\lambda$ is one-to-one. (Once you have completed this part of the problem, then you have proven that $G$ is isomorphic to a subgroup of the permutation group $\text{Sym}(G)$. This is called Cayley’s Theorem.)

2. Let $G$ be the group $D_3$ with elements id, $\rho$, $\rho^2$, $\sigma$, $\sigma \rho$, $\sigma \rho^2$. Compute the homomorphism $\lambda : D_3 \rightarrow \text{Sym}(D_3)$ as described in 1. It is natural to think of $\text{Sym}(D_3)$ as $S_6$ where the elements of $D_3$ are identified with $\{1, 2, 3, 4, 5, 6\}$ by $1 \leftrightarrow \text{id}$, $2 \leftrightarrow \rho$, $3 \leftrightarrow \rho^2$, $4 \leftrightarrow \sigma$, $5 \leftrightarrow \sigma \rho$, $6 \leftrightarrow \sigma \rho^2$, using the above order for the elements of $D_3$. For each $g$ in $D_3$, find the permutation $\lambda(g)$ in $S_6$.

3. Let $G$ be the group $Z_6$ with elements $1, 2, 3, 4, 5, 0$. Compute the homomorphism $\lambda : Z_6 \rightarrow \text{Sym}(Z_6)$ as described in 1. It is natural to think of $\text{Sym}(Z_6)$ as $S_6$ where the elements of $Z_6$ are identified with $\{1, 2, 3, 4, 5, 6\}$ by $1 \leftrightarrow 1$, $2 \leftrightarrow 2$, $3 \leftrightarrow 3$, $4 \leftrightarrow 4$, $5 \leftrightarrow 5$, $6 \leftrightarrow 6$, using the above order for the elements of $Z_6$. For each $g$ in $Z_6$, find the permutation $\lambda(g)$ in $S_6$.

4. The goal of this exercise is to prove that it makes sense to say “even permutation” or odd permutation”. Your book offers two proofs: one on 74-75 and one on 133-134. This exercise outlines the argument given on 133-134. Let $Z[x_1, \ldots, x_n]$ be the collection of polynomials in $n$ variables with integer coefficients. Notice that each element of $S_n$ gives a permutation of $Z[x_1, \ldots, x_n]$ by $\sigma(f(x_1, \ldots, x_n))$ is equal to $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Let $\Delta_n$ be the polynomial

$$\Delta_n = \prod_{1 \leq i < j \leq n}(x_i - x_j)$$

in $Z[x_1, \ldots, x_n]$.

(a) Let $\sigma$ and $\tau$ be in $S_n$ and $f \in Z[x_1, \ldots, x_n]$. Observe that $(\sigma \circ \tau)(f) = \sigma(\tau(f))$.

(b) Observe that if $\sigma \in S_n$, then $\sigma(\Delta_n)$ is equal to $\Delta_n$ or $-\Delta_n$. 

(c) Prove that if \( \sigma \in S_n \) is a transposition, then \( \sigma(\Delta_n) = -\Delta_n \).

(d) Prove that \( \phi: S_n \to \{1, -1\} \), given by \( \phi(\sigma) = \frac{\Delta_n}{\sigma(\Delta_n)} \) is a group homomorphism.

(e) What is the kernel of \( \phi \)?

5. Let \( r \) and \( s \) be distinct elements of \( \{1, 2, \ldots, n\} \), with \( n \geq 3 \). Prove that \( A_n \) is generated by the 3-cycles \( \{(rs)k | 1 \leq k \leq n, k \neq r, s\} \).

6. Let \( N \) be a normal subgroup of \( A_n \) for some \( n \geq 3 \). Suppose that \( N \) contains a 3-cycle. Prove that \( N = A_n \).

7. Prove that \( A_4 \) does not have any subgroups of order 6.

8. Fix \( n \geq 5 \). The purpose of this problem is to prove that \( A_n \) is a simple group, that is, that \( A_n \) does not contain any normal subgroups other than \( \{id\} \) and \( A_n \). (This result is a key step in the proof of Galois’s Theorem that there does not exist an algebraic formula which expresses the roots of an arbitrary fifth degree polynomial in terms of the coefficients.) We let \( N \) be a normal subgroup of \( A_n \), with \( N \neq \{id\} \). We will prove that \( N \) must equal \( A_n \).

(a) If \( N \) contains a three cycle, then prove that \( N = A_n \).

(b) Let \( \sigma = (a_1, a_2, \ldots, a_r) \tau \) be a decomposition into disjoint cycles with \( r \geq 4 \).
   Suppose that \( \sigma \) is in \( N \). Let \( \delta = (a_1, a_2, a_3) \). Calculate \( \sigma^{-1}(\delta \sigma \delta^{-1}) \).
   Prove that \( N = A_n \).

(c) Let \( \sigma = (a_1, a_2, a_3)(a_4, a_5, a_6) \tau \) be a decomposition into disjoint cycles.
   Suppose that \( \sigma \) is in \( N \). Let \( \delta = (a_1, a_2, a_4) \). Calculate \( \sigma^{-1}(\delta \sigma \delta^{-1}) \).
   Prove that \( N = A_n \).

(d) Let \( \sigma = (a_1, a_2, a_3) \tau \) be a decomposition into disjoint cycles, where \( \tau \) is a product of disjoint transpositions.
   Suppose that \( \sigma \) is in \( N \). Calculate \( \sigma^2 \).
   Prove \( N = A_n \).

(e) Let \( \sigma = (a_1, a_2)(a_3, a_4) \tau \) be a decomposition into disjoint cycles, where \( \tau \) is a product of disjoint transpositions.
   Suppose that \( \sigma \) is in \( N \). Let \( \delta = (a_1, a_2, a_3) \). Calculate \( \sigma^{-1}(\delta \sigma \delta^{-1}) \).
   Let \( b \) be an element of \( \{1, \ldots, n\} \) with \( b \) distinct from \( a_1, a_2, a_3 \). Let \( \xi = (a_1, a_3, b) \) and \( \zeta = (a_1, a_3)(a_2, a_4) \).
   Calculate \( \zeta(\xi \zeta \xi^{-1}) \).
   Prove \( N = A_n \).

(f) Make sure that we have covered all of the possibilities! Conclude that if \( n \geq 5 \),
then \( A_n \) is a simple group.