Homework October 1, 2004

1. Suppose that $H$ is a subgroup of the group $G$ and $ghg^{-1}$ is in $H$ for all $g \in G$ and $h \in H$.
   (a) Let $h_1$ be an arbitrary element of $H$ and $g$ be an arbitrary element of $G$.
       Prove that there exists an element $h$ of $H$ with $h_1 = ghg^{-1}$. (It is possible
       to give a proof which works for infinite groups as well as finite groups.)
   (b) Let $a, b, c$, and $d$ be elements of $G$ with $aH = bH$ and $cH = dH$. Prove
       that $acH = bdH$. (This is only a tiny extension of homework number 3 from
       September 29.)
   (c) Let $S$ be the set of cosets $S = \{aH \mid a \in G\}$ of $H$ in $G$. Part (b) shows that
       the operation on $S$ given by $(aH) * (bH) = abH$ is a well-defined function.
       Prove that $S$ is a group. (If you are looking for this somewhere, $S$ is usually
       written as $G/H$ and $S$ is called the “quotient group of $G$ mod $H$”, or the
       “factor group of $G$ mod $H$”. BY THE WAY: $S$ is not a subset of anything;
       we have to verify all of the axioms for group. Fortunately, this is very easy.)

2. (a) If $G$ is an abelian group and $H$ is a subgroup of $G$, then prove that $ghg^{-1}$
       is in $H$ for all $g \in G$ and $h \in H$.
   (b) If $G$ is a finite group with $2n$ elements and $H$ is a subgroup of $G$ with $n$
       elements, then prove that $ghg^{-1}$ is in $H$ for all $g \in G$ and $h \in H$.
   (b) If $G$ is a group and $H$ is a subgroup of the center of $G$, then prove that
       $ghg^{-1}$ is in $H$ for all $g \in G$ and $h \in H$.

3. Work out some examples of $G/H$ as described in problem 1c.
   (a) Let $G = D_4$ and $H = <\rho>$. Problem 2c tells us that it is legal to create
       $G/H$. What is this group? How many elements does it have? What is the
       multiplication table? Do you believe that this multiplication makes sense?
   (b) Let $G = D_4$ and $H = <\rho^2>$. Problem 2b tells us that it is legal to create
       $G/H$. What is this group? How many elements does it have? What is the
       multiplication table? Do you believe that this multiplication makes sense?
   (c) Let $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$. Problem 2a tells us that it is legal to create $G/H$.
       What is this group? How many elements does it have? What is the addition table?
       Do you believe that this addition makes sense? (Notice that the elements of
       this $G/H$ look like $a+H$ because the operation in $G$ is called + . Furthermore,
       the operation in $G/H$ is also called + ; that is, $(a + H) + (b + H) = a + b + H$.)