7. Let G be the group U_9. (a) LIST the elements of the set

$$S = \{g^3 \mid g \in G\}.$$

(b) Is S a subgroup of G? Justify your answer to (b).

Let $u = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$.

The elements of U_9 are $\{1, u, \ldots, u^8\}$

$(1)^3 = 1$
$(u)^3 = u^3$
$(u^2)^3 = u^6$
$(u^4)^3 = u^3$
$(u^5)^3 = u^6$
$(u^7)^3 = u^3$
$(u^8)^3 = u^6$
$(u^3)^3 = 1$
$(u^6)^3 = 1$

So $S = \{1, u^3, u^6\}$

S is a subgroup of G because S is closed $u^3 \cdot u^6 = 1$ and the inverse of each element of S is in S

or because of Problem 5

8. Give an example of a group G and subgroups H and K of G with $H \cup K$ not equal to a subgroup of G. Explain.

Let G be the subgroup $\{i0, \rho^1, \sigma\rho^2, \sigma\}$ of D_4

Let $H = \{i0, \rho^2\}$ ← we are given

$K = \{i0, \sigma\rho^2\}$ ←

$H \cup K = \{i0, \rho^2, \sigma\rho^2\}$ is not a group because this set is not closed $\rho^2 \in H \cup K$ $\sigma\rho^2 \in H \cup K$ but $\rho^2 \cdot \sigma\rho^2 = \sigma \notin H \cup K$.