Math 546, Exam 3, Fall, 2004

The exam is worth 50 points.

Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ...; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail.

I will leave your exam outside my office TOMORROW by about 6PM, you may pick it up any time between then and the next class.

I will post the solutions on my website at about 4:00 PM today.

1. (6 points) Define “normal subgroup”. Use complete sentences.
   The subgroup $N$ of the group $G$ is a normal subgroup if $gng^{-1} \in N$ for all $g \in G$ and $n \in N$.

2. (6 points) Define “cyclic group”. Use complete sentences.
   The group $G$ is a cyclic group if there exists an element $g \in G$ so that every element of $G$ has the form $g^n$ for some integer $n$.

3. (6 points) Define “generator”. Use complete sentences.
   The element $g$ of the group $G$ is a generator of $G$ if every element of $G$ has the form $g^n$ for some integer $n$.

4. (6 points) What is the order of $(1,1) + H$ in the group $\tilde{G} = \frac{G}{H}$, where $G = \mathbb{Z}_6 \times \mathbb{Z}_4$ and $H = \{(0,0), (3,0), (0,2), (3,2)\}$? Is $\tilde{G}$ a cyclic group? (A small amount of explanation is needed.)
   Let $x$ be the element $(1,1)$ of $G$. Observe that the least positive integer $n$ with $nx \in H$ is $n = 6$. Thus, the order of $x + H$ in $\tilde{G}$ is 6. The group $G$ has 6 elements, so $G$ IS cyclic. Here is our calculation:
   
   $x = (1,1) \notin H, \quad 2x = (2,2) \notin H, \quad 3x = (3,3) \notin H, \quad 4x = (4,4) \notin H, \quad 5x = (5,5) \notin H, \quad 6x = (6,6) = (0,2) \in H.$

5. (6 points) Find 2 distinct elements of order 2 in the group $\bar{G} = \frac{\mathbb{Z}_6 \times \mathbb{Z}_4}{<(2,2)>}$. Is $\bar{G}$ a cyclic group? (A small amount of explanation is needed.)
   Let $G$ be the group $\mathbb{Z}_6 \times \mathbb{Z}_4$, $H$ be the subgroup $<(2,2)>$ of $G$, $x$ be the element $(1,0)$ of $G$, and $y$ be the element $(0,1)$ of $G$. We see that
   
   $H = \{(0,0), (2,2), (4,0), (0,2), (2,0), (4,2)\}.$

   We also see that
   
   $x \notin H, \quad y \notin H, \quad x - y \notin H, \quad 2x \in H, \quad 2y \in H.$

   It follows that $x + H$ and $y + H$ are distinct elements of $\bar{G}$ of order 2.
6. (7 points)
(a) Find an element of order 3 in $\mathbb{Z}/3\mathbb{Z}$. (A small amount of explanation is needed.)
(b) Find an element of infinite order in $\mathbb{Z}/\mathbb{Z}$. (A small amount of explanation is needed.)

The coset $1/3 + \mathbb{Z}$ of $\mathbb{Z}/3\mathbb{Z}$ has order 3, because $1/3 \notin \mathbb{Z}$, $2/3 \notin \mathbb{Z}$, but $3/3 = 1 \in \mathbb{Z}$.

The coset $\pi + \mathbb{Z}$ of $\mathbb{Z}/\mathbb{Z}$ has infinite order because $n\pi \notin \mathbb{Z}$ for any positive integer $n$.

7. (7 points)
(a) Does there exist a function $\varphi: \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ with $\varphi(a + 9\mathbb{Z}) = a + 3\mathbb{Z}$ for all integers $a$? Explain thoroughly.
(b) Does there exist a function $\varphi: \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z}$ with $\varphi(a + 3\mathbb{Z}) = a + 9\mathbb{Z}$ for all integers $a$? Explain thoroughly.

(a) Yes, $\varphi$ IS a function. If the cosets $a + 9\mathbb{Z}$ and $b + 9\mathbb{Z}$ are equal, then $a - b \in 9\mathbb{Z} \subseteq 3\mathbb{Z}$; hence, the cosets $a + 3\mathbb{Z}$ and $b + 3\mathbb{Z}$ are equal.

(b) NO, $\varphi$ is NOT a function! We see that the cosets $0 + 3\mathbb{Z}$ and $3 + 3\mathbb{Z}$ are equal. We also see that the cosets $0 + 9\mathbb{Z}$ and $3 + 9\mathbb{Z}$ are NOT equal. There is no FUNCTION which sends $0 + 3\mathbb{Z}$ to $0 + 9\mathbb{Z}$ and $3 + 3\mathbb{Z}$ to $3 + 9\mathbb{Z}$.

8. (6 points) Let $K$ be a subgroup of the group $G$ and let $N$ be a normal subgroup of $G$. Prove that

$$H = \{kn \mid k \in K \text{ and } n \in N\}$$

is a subgroup of $G$.

**Closure:** Take $h_1$ and $h_2$ from $H$. We know that $h_1 = k_1n_1$ and $h_2 = k_2n_2$ for some $k_i \in K$ and $n_i \in N$. Observe that

$$h_1h_2 = k_1n_1k_2n_2 = k_1k_2(k_2^{-1}n_1k_2)n_2.$$ 

We know that $k_1k_2 \in K$ because $K$ is a group and $(k_2^{-1}n_1k_2)n_2 \in N$ because $N$ is a normal subgroup of $G$. Thus, $h_1h_2 \in H$ and $H$ is closed.

**Identity:** Let $\text{id}_G$ be the identity element of $G$, $\text{id}_K$ the identity element of $K$, and $\text{id}_N$ the identity element of $N$. We know $\text{id}_G = \text{id}_K\text{id}_N$ because all three identity elements are equal. Thus, the identity element of $G$ is in $H$.

**Inverses:** Take $h = kn$ from $H$, with $k \in K$ and $n \in N$. We know that $h^{-1} = n^{-1}k^{-1} = k^{-1}(kn^{-1}k^{-1})$. Furthermore, $k^{-1} \in K$ because $K$ is a group and $(kn^{-1}k^{-1}) \in N$ because $N$ is a normal subgroup of $G$. We conclude that $h^{-1}$ is in $H$.