Math 546, Exam 2, Fall, 2004

The exam is worth 50 points.

Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet. Take enough space for each problem. Turn in your solutions in the order: problem 1, problem 2, ...; although, by using enough paper, you can do the problems in any order that suits you.

If I know your e-mail address, I will e-mail your grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail.

I will leave your exam outside my office TOMORROW by about 5PM, you may pick it up any time between then and the next class.

I will post the solutions on my website at about 4:00 PM today.

1. (6 points) Define “subgroup”. Use complete sentences.

   The subset $H$ of the group $(G, *)$ is a subgroup of $G$, if $H$ is a group under the same operation $*$.  

2. (6 points) Define the “center of a group”. Use complete sentences.

   The center of the group $G$ is the set of all elements in $G$ which commute with every element in $G$.

3. (6 points) STATE Lagrange’s Theorem.

   If $H$ is a subgroup of the finite group $G$, then the order of $H$ divides the order of $G$.

4. (7 points) Let $G$ be a finite group with an even number of elements. Prove that there must exist an element $a \in G$ with $a \neq \text{id}$, but $a^2 = \text{id}$.

   Observe that $G$ is the disjoint union of the sets
   
   $$ Y = \{ g \in G \mid g^2 = \text{id} \} \quad \text{and} \quad N = \{ g \in G \mid g^2 \neq \text{id} \}. $$

   The set $Y$ always contains at least one element, namely $\text{id}$. Observe that if $g \in N$, then $g^{-1}$ is also in $N$ and $g \neq g^{-1}$. It follows that $N$ may be partitioned into a collection of subsets each of which consists of a pair of elements which are inverses of one another. Thus, $N$ contains an even number of elements. The hypothesis ensures that the group $G$ contains an even number of elements. We conclude that $Y$ contains an even number of elements. Since $Y$ contains at least one element, we now know that $Y$ must contain at least two elements. In other words, there does exist an element $g$ in $G$ with $g \neq \text{id}$, but $g^2 = \text{id}$.

5. (7 points) Give an example of a finite group $G$ and a proper subgroup $H$ of $G$, with $H$ not a cyclic group.

   Consider $H = \{ \text{id}, \sigma, \rho^2, \rho^2 \sigma \}$ and $G = D_4$. We have seen that $H$ is a subgroup of $G$. (If you like, $H$ is equal to the centralizer of $\sigma$.) It is clear $H$ is not cyclic because every element of $H$ squares to the identity element.
6. (6 points) Let $G$ be a group of order $pq$ where $p$ and $q$ are prime numbers. Prove that every proper subgroup of $G$ is cyclic.

If $H$ is a proper subgroup of $G$ and $H$ is larger than $\{\text{id}\}$, then Lagrange’s Theorem ensures that $H$ has order $p$ or $q$. Furthermore, Lagrange’s Theorem also ensures that every group of prime order is cyclic.

7. (6 points) Let $g$ be an element of the group $G$. Suppose that $G$ has order $n$. Prove that $g^n = \text{id}$.

Let $m$ equal the order of the cyclic subgroup $\langle g \rangle$ of $G$. It follows that $g^m = \text{id}$. Lagrange’s Theorem ensures that $m|n$; that is $n = mr$ for some integer $r$. We have $g^n = g^{mr} = (g^m)^r = (\text{id})^r = \text{id}$.

8. (6 points) Let $H$ be a subgroup of a group. Suppose that $g^{-1}hg \in H$ for all $g \in G$ and $h \in H$. Fix an element $g \in G$. Prove that $gH = Hg$, where $gH$ is the LEFT coset

$$gH = \{gh \mid h \in H\}$$

and $Hg$ is the RIGHT coset

$$Hg = \{hg \mid h \in H\}.$$