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4. What is the order of the element $\sqrt{2}/2 - i\sqrt{2}/2$ in the group $(\mathbb{C}^*, \times)$? Explain your answer.

$\omega = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ has order 8 because $\omega = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

$\omega^8 = 1$ and $\omega^n \neq 1$ for $1 \le n \le 7$.

5. Prove that every subgroup of $(\mathbb{Z}, +)$ is cyclic.

Let H be a subgroup of $\mathbb{Z}$. If $H = \{0\}$, we are finished. Hence forth, we assume that $\{0\} \subsetneq H$. Let h be the smallest positive integer in H. We will prove $\langle h \rangle = H$. It is obvious that $\langle h \rangle \subseteq H$. To finish the proof we show $H \subseteq \langle h \rangle$. Let x be any element of H. The division algorithm gives $x = hm + r$ for $m, r \in \mathbb{Z}$ with $0 \le r \le h-1$ but $r = hm - x \in H$. The definition of h gives $r = 0$. Thus $x = hm$ and $x \in \langle h \rangle$.