4. What is the order of the element \( \sqrt{2}/2 - i\sqrt{2}/2 \) in the group \((\mathbb{C}^*, \times)\)?

Explain your answer.

\[
\omega = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \text{ has order } 8 \text{ because } \omega = \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}
\]

\(\omega^8 = 1\) and \(\omega^n \neq 1\) for \(1 \leq n \leq 7\).

5. Prove that every subgroup of \((\mathbb{Z}, +)\) is cyclic.

Let \(H\) be a subgroup of \(\mathbb{Z}\). If \(H = \{0\}\) we are finished.

Henceforth, we assume that \(\{0\} \neq H\). Let \(h\) be the smallest positive integer in \(H\). We will prove \(H = \langle h \rangle\).

It is obvious that \(\langle h \rangle \subseteq H\). To finish the proof we show \(H \subseteq \langle h \rangle\). Let \(x\) be any element of \(H\).

The division algorithm gives \(x = km + r\) for \(m \in \mathbb{Z}\) with \(0 \leq r < h\), but \(r = km - x \in H\). The definition of \(h\) gives \(r = 0\). Thus \(x = km\) and \(x \in \langle h \rangle\).