PRINT Your Name: ____________________________

There are 6 problems on 3 pages. The exam is worth a total of 50 points. Problems 5 and 6 are worth 9 points each. The other problems are worth 8 points each. In this exam, a subgroup $H$ of a group $G$ is called proper if $H \subseteq G$.

1. TRUE or FALSE. (If true, PROVE it. If false, give a COUNTER EXAMPLE.)
   If every proper subgroup $H$ of a group $G$ is cyclic, then $G$ is cyclic.

   **False** The subgroup $\{e, 0, p^2, 0p^2\}$ of $D_4$ is not cyclic because every element squares to $e$. But the subgroups of this group all are cyclic because the product of any two of the non-identity elements is the third non-identity element.

2. TRUE or FALSE. (If true, PROVE it. If false, give a COUNTER EXAMPLE.)
   If $G$ is a cyclic group, then every proper subgroup $H$ of $G$ is cyclic.

   **True** Let $G = \langle g \rangle$. If $H = \langle e, 0^2 \rangle$, then $H$ is cyclic. Hence $H \cong \mathbb{Z}$.

   We assume $e, 0^2 \notin H$. Let $m$ be the least positive integer with $gm \notin H$. I claim $H \cong \mathbb{Z}/m\mathbb{Z}$.

   - If $H \cong \mathbb{Z}/m\mathbb{Z}$, then $H = \langle g \rangle$ for some $n$. Decide minimal $n$ to set $n = qm + r$ for integers $q$ and $r$ with $0 \leq r < m$.
   - We see that $g^n (gm)^{-q} \in H$. So $g^n \notin H$. The choice of $m$ tells us $r = 0$; hence $n = qm$ and $H = \langle g \rangle$. Thus $H \cong \mathbb{Z}/m\mathbb{Z}$. 
