Math 546, Exam 1, Spring 2010

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.
The exam is worth 50 points. There are **5** problems. Each problem is worth 10 points. Write **coherently** in **complete sentences**.
No Calculators or Cell phones.

1. Recall that U_6 is the subgroup $\{1, z, z^2, z^3, z^4, z^5\}$ of $(\mathbb{C}, \times)$, with $z = e^{\frac{2\pi\imath}{6}} = \cos(\frac{2\pi}{6}) + \imath\sin(\frac{2\pi}{6})$
   (a) Identify 2 subgroups of U_6 in addition to $\{1\}$ and U_6. (I don't need to see a proof.)
   (b) Which elements of U_6 generate U_6? (Recall that the element g of the group $(G, *)$ *generates* G if every element of G is equal to $\underbrace{g * g * \cdots * g}_{n \text{ times}}$, for some integer n.) (I do want to see an explanation.)

2. Recall that D_3 is the group $\{\mathrm{id}, \rho, \rho^2, \sigma, \sigma\rho, \sigma\rho^2\}$, where σ is reflection across the x-axis and ρ is rotation by $2\pi/3$ radians, counter-clockwise, fixing the origin.
   (a) Identify 4 subgroups of D_3 in addition to $\{\mathrm{id}\}$ and D_3. (I don't need to see a proof.)
   (b) Which elements of D_3 generate D_3? The word "generates" is defined in problem 1. (I do want to see an explanation.)

3. Let $(G, *)$ be a group and $H = \{g * g * g \mid g \in G\}$.
   (a) Assume that the group G is Abelian. Prove that H is a subgroup of G.
   (b) Give an example which shows that H is not always a subgroup of G. (Provide all details.)

4. Let $S = \mathbb{R} \setminus \{-2\}$. Define $*$ on S by $a * b = ab + 2a + 2b + 2$. Prove that $(S, *)$ is a group.

5. Recall that D_4 is the group $\{\mathrm{id}, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3\}$, where σ is reflection across the x-axis and ρ is rotation by $2\pi/4$ radians, counter-clockwise, fixing the origin. List the elements of the following set:

$$Z = \{\tau \in D_4 \mid \tau\omega = \omega\tau \text{ for all } \omega \text{ in } D_4\}.$$

   I do want an explanation. We saw in class that the 8 listed elements of D_4 are distinct; we also saw that $\rho\sigma = \sigma\rho^3$.