Math 546, Exam 1, Spring 2010

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **5** problems. Each problem is worth 10 points. Write **coherently** in **complete sentences**.

No Calculators or Cell phones.

1. **Recall that U_6 is the subgroup $\{1, z, z^2, z^3, z^4, z^5\}$ of $(\mathbb{C}, \times)$, with $z = e^{\frac{2\pi\imath}{6}} = \cos(\frac{2\pi}{6}) + \imath\sin(\frac{2\pi}{6})$.**
   (a) **Identify 2 subgroups of U_6 in addition to $\{1\}$ and U_6. (I don't need to see a proof.)**

   Two subgroups of U_6 are $\{1, z^2, z^4\}$ and $\{1, z^3\}$.

   (b) **Which elements of U_6 generate U_6? (Recall that the element g of the group $(G, *)$ *generates* G if every element of G is equal to $\underbrace{g * g * \cdots * g}_{n \text{ times}}$, for some integer n.) (I do want to see an explanation.)**

We see that z, and z^5 generate U_6. The powers of z^5 are: $z^5, z^4, z^3, z^2, z, 1$. The other elements all generate smaller subgroups of U_6 as is shown in (a).

2. **Recall that D_3 is the group $\{\text{id}, \rho, \rho^2, \sigma, \sigma\rho, \sigma\rho^2\}$, where σ is reflection across the x-axis and ρ is rotation by $2\pi/3$ radians, counterclockwise, fixing the origin.**
   (a) **Identify 4 subgroups of D_3 in addition to $\{\text{id}\}$ and D_3. (I don't need to see a proof.)**

   Four subgroups of D_3 are:

$$\{\text{id}, \sigma\}, \{\text{id}, \rho, \rho^2\}, \{id, \sigma\rho\}, \{\text{id}, \sigma\rho^2\}.$$

   (b) **Which elements of D_3 generate D_3? The word "generates" is defined in problem 1. (I do want to see an explanation.)**

   Not element of D_3 generates D_3. Indeed, id generates a subgroup with one element; σ, $\sigma\rho$, and $\sigma\rho^2$ each generate a subgroup with two elements; and ρ and ρ^2 each generate a subgroup with three elements.

3. **Let** $(G, *)$ **be a group and** $H = \{g * g * g \mid g \in G\}$ **.**

   (a) **Assume that the group** G **is Abelian. Prove that** H **is a subgroup of** G **.**

   Closure: Take a, b from H so $a = g * g * g$ for some $g \in G$ and $b = g' * g' * g'$ for some $g' \in G$. The group G is Abelian so

   $$a * b = (g * g * g) * (g' * g' * g') = (g * g') * (g * g') * (g * g'),$$

   which is in H because $g * g'$ is in G.

   Associativity: The operation $*$ is associative on all of G, so $*$ is associative on the subset H of G.

   Identity: If e is the identity element of G, then $e = e{*}e{*}e$ and therefore, $e \in H$.

   Inverses: Let $a \in G$. It follows that $a = g{*}g{*}g$ for some $g \in G$. Observe that g^{-1} is in G and therefore $g^{-1} * g^{-1} * g^{-1}$ is in H. Furthermore, $g^{-1} * g^{-1} * g^{-1}$ is the inverse of a because

   $$a * (g^{-1} * g^{-1} * g^{-1}) = (g * g * g) * (g^{-1} * g^{-1} * g^{-1}) = e$$

   and

   $$(g^{-1} * g^{-1} * g^{-1}) * a = (g^{-1} * g^{-1} * g^{-1}) * (g * g * g) = e.$$

   (b) **Give an example which shows that** H **is not always a subgroup of** G **. (Provide all details.)**

   If G is the group D_3, then

   $$H = \{\mathrm{id}, \sigma, \sigma\rho, \sigma\rho^2\}$$

   because,

   $$\mathrm{id}^3 = \mathrm{id}, \rho^3 = \mathrm{id}, (\rho^2)^3 = \mathrm{id}, \sigma^3 = \sigma, (\sigma\rho)^3 = (\sigma\rho), (\sigma\rho^2)^3 = \sigma\rho^2.$$

   The set H is not a group because it is not closed. Indeed, we see that σ and $\sigma\rho$ are both in H but

   $$\sigma(\sigma\rho) = \rho \notin H.$$

4. **Let $S = \mathbb{R} \setminus \{-2\}$. Define $*$ on S by $a * b = ab + 2a + 2b + 2$. Prove that $(S, *)$ is a group.**

Closure: Take a, b from S. We must show that $a * b$ is in S. Well, $a * b = ab + 2a + 2b + 2$, which is clearly a real number. We must check that $ab + 2a + 2b + 2$ is not equal to -2. If $ab + 2a + 2b + 2$ were equal to -2, then $ab + 2a + 2b + 2 = -2$; so, $ab + 2a + 2b + 4 = 0$; that is, $(a + 2)(b + 2) = 0$; so $a = -2$ or $b = -2$. On the other hand, a and b are in S; so neither a nor b is -2. We conclude that $ab + 2a + 2b + 2 \neq -2$; therefore, $ab + 2a + 2b + 2 \in S$.

Associativity: Take a, b, and c from S. Observe that

$$a * (b * c) = a * (bc + 2b + 2c + 2) = a(bc + 2b + 2c + 2) + 2a + 2(bc + 2b + 2c + 2) + 2$$

$$= abc + 2(ab + ac + bc) + 4(a + b + c) + 6.$$

On the other hand,

$$(a * b) * c = (ab + 2a + 2b + 2) * c = (ab + 2a + 2b + 2)c + 2(ab + 2a + 2b + 2) + 2c + 2$$

$$= abc + 2(ab + ac + bc) + 4(a + b + c) + 6.$$

We see that $a * (b * c) = (a * b) * c$.

Identity: The number -1 is the identity element of S because $a * (-1) = a(-1) + 2a + 2(-1) + 2 = a$ and $(-1) * a = (-1)a + 2(-1) + 2a + 2 = a$ for all $a \in S$.

Inverses: Take $a \in S$. The inverse of a is $\frac{-3-2a}{a+2}$ because

$$a * \frac{-3 - 2a}{a + 2} = a\frac{-3 - 2a}{a + 2} + 2a + 2\frac{-3 - 2a}{a + 2} + 2 = \frac{(a + 2)(-3 - 2a)}{a + 2} + 2a + 2 =$$

$$-3 - 2a + 2a + 2 = -1.$$

The operation * is commutative; so, $\frac{-3-2a}{a+2} * a$ is also equal to 0. Notice, also, that $\frac{-3-2a}{a+2} \in S$ because $\frac{-3-2a}{a+2}$ is a real number (since $a \neq -2$) and $\frac{-3-2a}{a+2}$ is not equal to -2; because if $\frac{-3-2a}{a+2}$ were equal to -2, then $\frac{-3-2a}{a+2} = -2$, so $-3 - 2a = -2a - 4$; that is, $-3 = -4$.

5. **Recall that** D_4 **is the group** $\{\mathbf{id}, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3\}$ **, where** σ **is reflection across the** x **-axis and** ρ **is rotation by** $2\pi/4$ **radians, counter-clockwise, fixing the origin. List the elements of the following set:**

$$Z = \{\tau \in D_4 \mid \tau\omega = \omega\tau \textbf{ for all } \omega \textbf{ in } D_4\}.$$

I do want an explanation. We saw in class that the 8 **listed elements of** D_4 **are distinct; we also saw that** $\rho\sigma = \sigma\rho^3$ **.**

We compute that $Z = \boxed{\{\text{id}, \rho^2\}}$. First of all, it is clear that id commutes with all elements of D_4, so $\text{id} \in Z$. It is also easy to see that ρ^2 commutes with every element of Z. The key to establishing this assertion is:

$$\rho^2\sigma = \rho\sigma\rho^3 = \sigma\rho^3\rho^3 = \sigma\rho^2.$$

So, now we have

$$\rho^2(\sigma^i\rho^j) = \sigma^i\rho^2\rho^j = (\sigma^i\rho^j)\rho^2$$

for all i and j. So ρ^2 commutes with all elements of D_4.

On the other hand, none of the other elements of D_4 are in Z. Indeed, $\rho\sigma = \sigma\rho^3 \neq \sigma\rho$; so neither σ nor ρ is in Z. Also,

$$\rho^3(\sigma\rho) = \sigma\rho^3\rho^3\rho^3\rho = \sigma\rho^2 \neq \sigma = (\sigma\rho)\rho^3;$$

thus, neither ρ^3 nor $\sigma\rho$ is in Z. Also,

$$(\sigma\rho^3)\sigma = \sigma\sigma\rho^3\rho^3\rho^3 = \rho \neq \rho^3 = \sigma(\sigma\rho^3);$$

thus, $\sigma\rho^3 \notin Z$. Finally,

$$\rho(\sigma\rho^2) = \sigma\rho^3\rho^2 = \sigma\rho \neq \sigma\rho^3 = (\sigma\rho^2)\rho;$$

so, $\sigma\rho^2 \notin Z$.