Math 546, Exam 1, Fall 2011

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **5** problems. Each problem is worth 10 points. Write **coherently** in **complete sentences**.

No Calculators or Cell phones.

1. Recall that U_{12} is the subgroup $\{1, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}, z^{11}\}$ of $(\mathbb{C} \setminus \{0\}, \times)$, with $z = e^{\frac{2\pi\imath}{12}} = \cos(\frac{2\pi}{12}) + \imath \sin(\frac{2\pi}{12})$.
   (a) Identify 4 subgroups of U_{12} in addition to $\{1\}$ and U_{12}. Please give a complete explanation.
   (b) Which elements of U_{12} generate U_{12}? (Recall that the element g of the group $(G, *)$ *generates* G if every element of G is equal to $\underbrace{g * g * \cdots * g}_{n \text{ times}}$, for some integer n.) Please give a complete explanation.

2. Let $S = \mathbb{R} \setminus \{4\}$. Define $*$ on S by $a * b = 20 - 4a - 4b + ab$. Prove that $(S, *)$ is a group.

3. Let G be a group with identity element id. Suppose that H and K are subgroups of G with $H \neq \{\text{id}\}$ and $K \neq \{id\}$. Is it possible for $H \cap K$ to equal $\{\text{id}\}$? If $H \cap K = \{\text{id}\}$ is possible, then give an example. If $H \cap K = \{\text{id}\}$ is not possible, then give a proof. (Recall that $H \cap K$ is the *intersection* of H and K; that is, $H \cap K = \{g \in G \mid g \in H \text{ AND } g \in K\}$.)

4. Let G be a group. Suppose that H and K are subgroups of G. Does $H \cup K$ have to be a subgroup of G? If $H \cup K$ is always a subgroup of G, then give a proof. If it is possible for $H \cup K$ not to be a subgroup of G, then give an example. (Recall that $H \cup K$ is the *union* of H and K; that is, $H \cup K = \{g \in G \mid g \in H \text{ OR } g \in K\}$.)

5. Let $(G, *)$ be an Abelian group with identity element id and $H = \{g \in G \mid g * g * g = \text{id}\}$. Prove that H is a subgroup of G.