Let $W$ be a subspace of $\mathbb{R}^n$ and let $A$ be an $m \times n$ matrix. Define $V$ to be the following subset of $\mathbb{R}^m$:

$$
V = \{ y \in \mathbb{R}^m \mid y = Ax \text{ for some } x \text{ in } W \}.
$$

Prove that $V$ is a subspace of $\mathbb{R}^m$.

**ANSWER:**

**The set $V$ is closed under addition:** Take arbitrary elements $y_1$ and $y_2$ of $V$. The definition of $V$ tells us that there exist $x_1$ and $x_2$ in $W$ with $y_1 = Ax_1$ and $y_2 = Ax_2$. We see that $y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2)$. We know that $x_1 + x_2$ is in $W$, because $W$ is a vector space. Thus, $y_1 + y_2$ is in $V$.

**The set $V$ is closed under scalar multiplication:** Keep the arbitrary vector $y_1 \in V$ from above. Let $r$ be an arbitrary real number. We see that $ry_1 = rAx_1 = A(rx_1)$. The vector $rx_1$ is in $W$, because $W$ is a vector space; and therefore, $ry_1$ is in $V$.

**The set $V$ contains the zero vector:** The zero vector $0$ of $\mathbb{R}^m$ is in the vector space $W$; therefore $0 = A0$ is in $V$. 