3. (10 points) Define "basis".

The vectors \(v_1, \ldots, v_p\) are a basis for the vector space if
1. \(v_1, \ldots, v_p\) are linearly independent and
2. \(v_1, \ldots, v_p\) span the space.

4. (10 points) True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If \(A\) and \(B\) are \(2 \times 2\) matrices with \(A\) non-singular, then the column space of \(AB\) is equal to the column space of \(B\).

**False**

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

The column space of \(B\) is \(\{[a] | a \in \mathbb{R}\}\)

The column space of \(AB\) is \(\{[a] | a \in \mathbb{R}\}\)

5. (10 points) True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If \(A\) and \(B\) are \(2 \times 2\) matrices with \(A\) non-singular, then the null space of \(AB\) is equal to the null space of \(B\).

**True**

The null space of \(AB\) is equal to the null space of \(A\) where

\[ \text{If } x \in \text{null space } AB \text{ then } ABx = 0 \text{ so } \exists x \in \text{null space } A \]

For us \(\text{null space } AB \subseteq \text{null space } B\)

\[ \text{If } x \in \text{null space } AB \text{ then } ABx = 0 \text{ multiply both sides on the left by } A^{-1} \text{ so } A^{-1}ABx = A^{-1}0 \text{ i.e. } Bx = 0 \]

So \(x \in \text{null space } B\)