Math 544, Exam 3, Summer 2007

Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are 6 problems on TWO sides. The exam is worth a total of 50 points. SHOW your work. CIRCLE your answer. CHECK your answer whenever possible. No Calculators.

If I know your e-mail address, I will e-mail your grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail.

You should KEEP this copy of your exam.

I will post the solutions on my website sometime after 3:15 today.

1. (7 points) Let \( V = \{ p(x) \in \mathcal{P}_3 \mid \int_0^1 p(x)dx = 0 \} \). Is \( V \) a vector space? If yes, then find a basis for \( V \). If no, then show why not? (Recall that \( \mathcal{P}_3 \) is the vector space of polynomials of degree less than or equal to 3.)

2. (7 points) Let \( V \) be the set of singular 2 \( \times \) 2 matrices. Is \( V \) a vector space? If yes, then find a basis for \( V \). If no, then show why not?

3. (7 points) Let \( U \subseteq V \) be vector spaces. Suppose that \( v_1, v_2, v_3, v_4 \) is a basis for \( V \). Suppose further that \( v_1 \) and \( v_2 \) are in \( U \), but \( v_3 \) and \( v_4 \) are not in \( U \). What is the dimension of \( U \)? Explain your answer VERY THOROUGHLY.

4. (7 points) Let \( A \) be a 3 \( \times \) 5 matrix. Suppose that \( z_1, z_2, x_1, \) and \( x_2 \) are vectors in \( \mathbb{R}^5 \) and \( y_1 \) and \( y_2 \) are vectors in \( \mathbb{R}^3 \). Suppose further that \( Az_1 = 0, Az_2 = 0, Ax_1 = y_1, \) and \( Ax_2 = y_2 \). Suppose finally, that \( z_1 \) and \( z_2 \) are linearly independent, and that \( y_1 \) and \( y_2 \) are linearly independent. Do \( z_1, z_2, x_1, x_2 \) have to be linearly independent? If yes, give a complete proof. If no, give a counter example.
5. (11 points) In this problem, if $M$ is a matrix, then let $\mathcal{I}(M)$ denote the column space of $M$. Let $A$ and $B$ be $n \times n$ matrices. Answer each question. If the answer is yes, then give a proof. If the answer is no, then give a counter example.

(a) Is $\mathcal{I}(A)$ always a subset of $\mathcal{I}(AB)$?
(b) Is $\mathcal{I}(B)$ always a subset of $\mathcal{I}(AB)$?
(c) Is $\mathcal{I}(AB)$ always a subset of $\mathcal{I}(A)$?
(d) Is $\mathcal{I}(AB)$ always a subset of $\mathcal{I}(B)$?
(e) Suppose $B$ is non-singular. Is $\mathcal{I}(A)$ always a subset of $\mathcal{I}(AB)$?
(f) Suppose $B$ is non-singular. Is $\mathcal{I}(B)$ always a subset of $\mathcal{I}(AB)$?
(g) Suppose $B$ is non-singular. Is $\mathcal{I}(AB)$ always a subset of $\mathcal{I}(A)$?
(h) Suppose $B$ is non-singular. Is $\mathcal{I}(AB)$ always a subset of $\mathcal{I}(B)$?

6. (11 points) Let $A = \begin{bmatrix} 1 & 7 & 8 & 5 & 1 & 8 & 7 & 2 & 11 \\ 1 & 7 & 8 & 5 & 2 & 14 & 11 & 2 & 11 \\ 1 & 7 & 8 & 5 & 2 & 14 & 11 & 3 & 13 \\ 3 & 21 & 24 & 15 & 5 & 36 & 29 & 7 & 35 \end{bmatrix}$. Find a basis for the null space of $A$. Find a basis for the column space of $A$. Find a basis for the row space of $A$. Express each column of $A$ as a linear combination of the basis you have chosen for the column space of $A$. Express each row of $A$ as a linear combination of the basis you have chosen for the row space of $A$. 