Math 544, Exam 2, Summer 2007, Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are 9 problems on TWO sides. The exam is worth a total of 50 points. SHOW your work. CIRCLE your answer. CHECK your answer whenever possible. No Calculators.

If I know your e-mail address, I will e-mail your grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail.

cc

You should KEEP this copy of your exam.

I will post the solutions on my website sometime after 3:15 today.

1. (6 points) Let $A$ be a fixed $n \times n$ matrix and let $W = \{ x \in \mathbb{R}^n | Ax = 2x \}$. Is $W$ a vector space? If yes, explain why. If no, give an example to show that the rules of vector space do not hold.

YES! We see that $W$ is the null space of the matrix $A - 2I_n$. The null space of every matrix is a vector space.

2. (6 points) Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 | |x_1| = |x_2| \right\}$. Is $W$ a vector space? If yes, explain why. If no, give an example to show that the rules of vector space do not hold.

NO! The vectors $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are in $W$, but $w_1 + w_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is not in $W$.

3. (6 points) Define “null space”. Use complete sentences. Include everything that is necessary, but nothing more.

The null space of the matrix $A$ is the set of all column vectors $x$ with $Ax = 0$.

4. (6 points) Define “non-singular”. Use complete sentences. Include everything that is necessary, but nothing more.

The $n \times n$ matrix $A$ is non-singular if the only vector $x$ in $\mathbb{R}^n$ with $Ax = 0$ is $x = 0$. 
5. (6 points) Let $A$ be an $n \times n$ matrix. List three statements that are equivalent to the statement "$A$ is non-singular". Do not repeat your answer to problem 4.

1. The columns of $A$ are linearly independent.
2. The system of equations $Ax = b$ has a unique solution for every $b$ in $\mathbb{R}^n$.
3. The matrix $A$ is invertible.

6. (5 points) Let $A$ and $B$ be symmetric $n \times n$ matrices. Does the matrix $AB$ HAVE to be symmetric? If yes, PROVE the statement. If no, give an EXAMPLE.

NO! The matrices $A = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are both symmetric; but the matrix $AB = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ is not symmetric.

7. (5 points) Let $v_1, v_2, v_3, v_4$ be vectors in $\mathbb{R}^5$. Suppose that $v_1, v_2, v_3, v_4$ are linearly independent. Do the vectors $v_1, v_2, v_3$ HAVE to be linearly independent? If yes, PROVE the result. If no, show an EXAMPLE.

YES! Suppose that $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. It follows that

$$(*) \quad c_1 v_1 + c_2 v_2 + c_3 v_3 + 0v_4 = 0.$$ 

The vectors $v_1, v_2, v_3, v_4$ are linearly independent; hence the only coefficients which cause $(*)$ to happen are $c_1 = c_2 = c_3 = 0$. We conclude that $v_1, v_2, v_3$ are linearly independent.

8. (5 points) Let $v_1, v_2, v_3$ be non-zero vectors in $\mathbb{R}^4$. Suppose that $v_i^T v_j = 0$ for all subscripts $i$ and $j$ with $i \neq j$. Prove that $v_1, v_2, v_3$ are linearly independent.

Suppose $c_1, c_2,$ and $c_3$ are numbers with

$$(*) \quad c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.$$ 

Multiply by $v_1^T$ to get

$$c_1 v_1^T v_1 + c_2 v_1^T v_2 + c_3 v_1^T v_3 = 0.$$ 

The hypothesis tells us that $v_1^T v_2 = 0$ and $v_1^T v_3 = 0$. So, $c_1 v_1^T v_1 = 0$. The hypothesis also tells us that $v_1$ is not zero; from which it follows that $v_1^T v_1 \neq 0$. We conclude that $c_1 = 0$. Multiply $(*)$ by $v_2^T$ to see that $c_2 \cdot v_2^T v_2 = 0$; hence, $c_2 = 0$, since the number $v_2^T v_2 \neq 0$. Multiply $(*)$ by $v_3^T$ to conclude that $c_3 = 0$. We have shown that each $c_i$ MUST be zero. We conclude that $v_1, v_2$, and $v_3$ are linearly independent.
9. (5 points) Consider the vectors

\[ w = \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \]

Is the vector \( w \) in the span of the vectors \( v_1 \), \( v_2 \), and \( v_3 \)? Explain thoroughly.

The question asks if \( Ax = w \) has a solution where

\[ A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}. \]

We apply the technique of Gaussian Elimination to

\[ \begin{bmatrix} 1 & 4 & 7 & | & 7 \\ 2 & 5 & 8 & | & 8 \\ 3 & 6 & 9 & | & 10 \end{bmatrix} \]

Replace \( R2 \leftrightarrow R2 - 2R1 \) and \( R3 \leftrightarrow R3 - 3R1 \) to obtain

\[ \begin{bmatrix} 1 & 4 & 7 & | & 7 \\ 0 & -3 & -6 & | & -6 \\ 0 & -6 & -12 & | & -11 \end{bmatrix} \]

Replace \( R3 \leftrightarrow R3 - 2R2 \) to obtain

\[ \begin{bmatrix} 1 & 4 & 7 & | & 7 \\ 0 & -3 & -6 & | & -6 \\ 0 & 0 & 0 & | & 1 \end{bmatrix} \]

The bottom row shows that \( Ax = w \) does not have a solution. It follows that \( w \) is NOT in the span of the vectors \( v_1 \), \( v_2 \), and \( v_3 \).