Math 544, Exam 1, Summer 2007 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

Please leave room in the upper left corner for the staple.

There are 7 problems. The exam is worth a total of 50 points. SHOW your work. CIRCLE your answer. CHECK your answer whenever possible. No Calculators.

If I know your e-mail address, I will e-mail your grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail.

I will post the solutions on my website sometime after 3:15 today.

1. (7 points) Define “linearly independent”. Use complete sentences. Include everything that is necessary, but nothing more.

The vectors \( v_1, \ldots, v_p \) in \( \mathbb{R}^n \) are linearly independent if the only numbers \( c_1, \ldots, c_p \) with \( \sum_{i=1}^{p} c_i v_i = 0 \) are \( c_1 = c_2 = \cdots = c_p = 0 \).

2. (7 points) Define “non-singular”. Use complete sentences. Include everything that is necessary, but nothing more.

The square matrix \( A \) is non-singular if the only column vector \( x \) with \( Ax = 0 \) is \( x = 0 \).

3. (7 points) Let \( A \) be an \( n \times n \) matrix. List two conditions which are equivalent to the statement “\( A \) is non-singular”. Do not repeat your answer to problem 2.

The following conditions are equivalent to the statement “\( A \) is non-singular”.

(1) The columns of \( A \) are linearly independent.

(2) The system of equations \( Ax = b \) has a unique solution for each vector \( b \in \mathbb{R}^n \).

4. (8 points) Find the GENERAL solution of the following system of linear equations. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.

\[
\begin{align*}
x_1 + 3x_2 + 4x_3 + 2x_4 + 4x_5 &= 16 \\
x_1 + 3x_2 + 4x_3 + 3x_4 + 6x_5 &= 21 \\
2x_1 + 6x_2 + 8x_3 + 5x_4 + 10x_5 &= 37
\end{align*}
\]
Consider
\[
\begin{bmatrix}
1 & 3 & 4 & 2 & 4 & | & 16 \\
1 & 3 & 4 & 3 & 6 & | & 21 \\
2 & 6 & 8 & 5 & 10 & | & 37
\end{bmatrix}
\]

Apply \( R_2 \mapsto R_2 - R_1 \) and \( R_3 \mapsto R_3 - 2R_1 \) to get:
\[
\begin{bmatrix}
1 & 3 & 4 & 2 & 4 & | & 16 \\
0 & 0 & 0 & 1 & 2 & | & 5 \\
0 & 0 & 0 & 1 & 2 & | & 5
\end{bmatrix}
\]

Apply \( R_3 \mapsto R_3 - R_2 \) and \( R_1 \mapsto R_1 - 2R_2 \) to get:
\[
\begin{bmatrix}
1 & 3 & 4 & 0 & 0 & | & 6 \\
0 & 0 & 0 & 1 & 2 & | & 5 \\
0 & 0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
\]

The general solution of this system of equations is
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

Some specific solutions are:
\[
\begin{bmatrix}
6 \\
0 \\
0 \\
5 \\
0
\end{bmatrix}, \begin{bmatrix}
3 \\
1 \\
1 \\
5 \\
0
\end{bmatrix}, \begin{bmatrix}
2 \\
0 \\
0 \\
5 \\
0
\end{bmatrix}, \begin{bmatrix}
6 \\
0 \\
0 \\
0 \\
3
\end{bmatrix}
\]

(We took \( x_2 = x_3 = x_5 = 0 \), \( x_2 = 1 \) with \( x_3 = x_5 = 0 \), \( x_3 = 1 \) with \( x_2 = x_5 = 0 \), \( x_5 = 1 \) with \( x_2 = x_3 = 0 \).) We check that each specific solution does indeed satisfy the equations:
\[
\begin{align*}
6 + 10 &= 16 & 3 + 3 + 10 &= 16 & 2 + 4 + 10 &= 16 \\
6 + 15 &= 21 & 3 + 3 + 15 &= 21 & 2 + 4 + 15 &= 21 \\
12 + 25 &= 37 & 6 + 6 + 25 &= 37 & 4 + 8 + 25 &= 37 \\
6 + 6 + 4 &= 16 \\
6 + 9 + 6 &= 21 \\
12 + 15 + 10 &= 37.
\end{align*}
\]
5. (7 points) Consider the system of linear equations.

\[\begin{align*}
x_1 + 4ax_2 &= 4 \\
ax_1 + x_2 &= 2.
\end{align*}\]

(a) Which values for \( a \) cause the system to have no solution?
(b) Which values for \( a \) cause the system to have exactly one solution?
(c) Which values for \( a \) cause the system to have an infinite number of solutions?

Explain thoroughly.

Apply row operations to

\[
\begin{bmatrix}
1 & 4a & 4 \\
a & 1 & 2
\end{bmatrix}
\]

Apply \( R_2 \mapsto R_2 - aR_1 \) to get

\[
\begin{bmatrix}
1 & 4a & 4 \\
0 & 1 - 4a^2 & 2 - 4a
\end{bmatrix}
\]

If \( 1 - 4a^2 \neq 0 \), then the system of equations has a unique solution.
If \( 1 - 4a^2 = 0 \) and \( 2 - 4a = 0 \), then the system of equations has infinitely many solutions.
If \( 1 - 4a^2 = 0 \) and \( 2 - 4a \neq 0 \), then the system of equations has no solution.
We notice that \( 1 - 4a^2 = 0 \) when \((1 - 2a)(1 + 2a) = 0\). Thus, \( a = \frac{1}{2} \), or \( a = -\frac{1}{2} \).
We notice that when \( a = \frac{1}{2} \), then \( 2 - 4a = 0 \)
We notice that when \( a = -\frac{1}{2} \), then \( 2 - 4a \neq 0 \).
We conclude

| If \( a \) is different than \( \frac{1}{2} \) and \( -\frac{1}{2} \), then the system has a unique solution. |
| If \( a = \frac{1}{2} \), then the system has infinitely many solutions. |
| If \( a = -\frac{1}{2} \), then the system has no solution. |

6. (7 points) Are the vectors

\[
v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}
\]

linearly independent? Explain thoroughly.
These vectors are linearly dependent because \( v_1 - 2v_2 + v_3 = 0 \).

7. (7 points) Let \( v_1, v_2, v_3, v_4 \) be vectors in \( \mathbb{R}^5 \). Suppose that \( v_1, v_2, v_3, v_4 \) are linearly dependent. Do the vectors \( v_1, v_2, v_3 \) HAVE to be linearly dependent? If yes, PROVE the result. If no, show an EXAMPLE.

No. \[
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1 \\
1 \\
0
\end{bmatrix}
\] are linearly dependent, but \( v_1, v_2, v_3 \) are linearly independent.