Math 544, Exam 2, Summer 2006 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet. Leave room on the upper left hand corner of each page for the staple. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

The exam is worth a total of 50 points.

SHOW your work. CIRCLE your answer. CHECK your answer whenever possible. No Calculators.

If I know your e-mail address, I will e-mail your grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail.

If you would like, I will leave your graded exam outside my office door. You may pick it up any time before the next class. If you are interested, be sure to tell me.

I will post the solutions on my website shortly after class is finished.

1. (10 points) Let

\[
A = \begin{bmatrix}
1 & 2 & 3 & 1 & 1 & 3 \\
2 & 4 & 6 & 2 & 1 & 5 \\
2 & 4 & 6 & 1 & 2 & 5 \\
2 & 4 & 6 & 1 & 1 & 4 \\
3
\end{bmatrix}.
\]

We apply Guassian Elimination to the matrix \(A\). Apply \(R_2 \leftrightarrow R_2 - 2R_1\), \(R_3 \leftrightarrow R_3 - 2R_1\), and \(R_4 \leftrightarrow R_4 - 2R_1\) to obtain

\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 & -2 \\
1
\end{bmatrix}.
\]

Exchange rows 2 and 3 to obtain

\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 1 & 3 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 & -2 \\
1
\end{bmatrix}.
\]
Apply $R_1 \leftrightarrow R_1 + R_2$ and $R_4 \leftrightarrow R_4 - R_2$ to obtain
\[
\begin{bmatrix}
1 & 2 & 3 & 0 & 1 & 2 \\
0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & -1 & -1
\end{bmatrix}.
\]
Apply $R_1 \leftrightarrow R_1 + R_3$ and $R_4 \leftrightarrow R_4 - R_3$ to obtain
\[
\begin{bmatrix}
1 & 2 & 3 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
Multiply rows 2 and 3 by $-1$ to obtain
\[
\begin{bmatrix}
1 & 2 & 3 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
(a) Find a basis for the null space of $A$.

The vectors
\[
w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]
are a basis for the null space of $A$.

(b) Find a basis for the column space of $A$.

The vectors
\[
A_{s,1} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad A_{s,4} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad A_{s,5} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}
\]
are a basis for the column space of $A$. Notice that I am writing $A_{*,j}$ for column $j$ of the matrix $A$.

(c) Find a basis for the row space of $A$.

The vectors $z_1 = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}$, $z_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$, and $z_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ are a basis for the row space of $A$.

(d) Express each column of $A$ in terms of your answer to (b).

We see that $A_{*,2} = 2A_{*,1}$, $A_{*,3} = 3A_{*,1}$, $A_{*,6} = A_{*,1} + A_{*,4} + A_{*,5}$.

(e) Express each row of $A$ in terms of your answer to (c).

I write $A_{i,*}$ for row $i$ of $A$. We see that $A_{1,*} = z_1 + z_2 + z_3$, $A_{2,*} = 2z_1 + 2z_2 + z_3$, $A_{3,*} = 2z_1 + z_2 + 2z_3$, and $A_{4,*} = 2z_1 + z_2 + z_3$.

2. (8 points) Find an orthogonal basis for the null space of $A = \begin{bmatrix} 1 & 1 & 1 & 2 \end{bmatrix}$.

We start with the basis $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.
for the null space of $A$. Let $u_1 = v_1$. Let

$$u'_2 = v_2 - \frac{u_1^T v_2}{u_1^T u_1} u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}.$$ 

Let

$$u_2 = 2u'_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}.$$ 

We check that $Au_2 = 0$ and $u_1^T u_2 = 0$. Let

$$u'_3 = v_3 - \frac{u_1^T v_3}{u_1^T u_1} u_1 - \frac{u_2^T v_3}{u_2^T u_2} u_2 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{6} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}.$$ 

Let

$$u_3 = 3u'_3 = \begin{bmatrix} -2 \\ -2 \\ -2 \\ 3 \end{bmatrix}.$$ 

We check that $Au_3 = 0$, $u_1^T u_3 = 0$, and $u_2^T u_3 = 0$. We conclude that

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -2 \\ -2 \\ -2 \\ 3 \end{bmatrix}$$

is an orthogonal basis for the null space of $A$. 
3. (8 points) Let $A$ and $B$ be $n \times n$ matrices with $A$ non-singular. For each question: If the answer is yes, then PROVE the assertion. If the answer is no, then give a COUNTER EXAMPLE.

(a) Does the null space of $BA$ have to equal the null space of $B$?

NO. Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We see that $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The matrices $B$ and $AB$ have different null spaces. For example, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in the null space of $B$, but not the null space of $BA$. On the other hand, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the null space of $BA$ but not the null space of $B$.

(b) Does the column space of $BA$ have to equal the column space of $B$?

YES. The column space of $BA$ is contained in the column space of $B$ because every vector in the column space of $BA$ has the form $BAv$ for some vector $v$. It is clear that $BAv = B(Av)$ which is $B$ times a vector and therefore in the column space of $B$.

The fact that the column space of $B$ is contained in the column space of $BA$ uses the hypothesis that $A$ is non-singular. A typical element of the column space of $B$ has the form $Bu$ for some vector $u$. We see that $Bu = BA(A^{-1}u)$ and this vector has the form $BA$ times some vector; so this vector is also in the column space of $BA$.

(d) Does the dimension of column space of $BA$ have to equal the dimension of column space of $B$?

YES. The matrices $B$ and $BA$ have the SAME column space. This one column space has a dimension.

(c) Does the dimension of the null space of $BA$ have to equal the dimension of the null space of $B$?

YES. The nullity of $B$ is equal to the number of columns of $B$ minus the rank of $B$. The nullity of $BA$ is equal to the number of columns of $BA$ minus the rank of $BA$. The matrices $B$ and $BA$ have the same rank by part (d). The matrix $A$
is square; so, $B$ and $BA$ have the same number of columns. We conclude that $B$ and $BA$ have the same nullity.

4. (3 points) Define “null space”. Use complete sentences. Include everything that is necessary, but nothing more.

The null space of the matrix $A$ is the set of all vectors $v$ such that $Av = 0$.

5. (3 points) Define “column space”. Use complete sentences. Include everything that is necessary, but nothing more.

The column space of the matrix $A$ is the set of all vectors $Av$ for some vector $v$.

6. (3 points) Define “dimension”. Use complete sentences. Include everything that is necessary, but nothing more.

The dimension of the vector space $V$ is the number of vectors in a basis for $V$.

7. (5 points) Suppose $U \subseteq V$ are vector spaces with the same finite dimension. Does $U$ have to equal $V$? If the answer is yes, then PROVE the assertion. If the answer is no, then give a COUNTER EXAMPLE.

YES. Let $n = \dim U = \dim V$. Let $u_1, \ldots, u_n$ be a basis for $V$. The vectors $u_1, \ldots, u_n$ are linearly independent vectors in $V$. One of the dimension theorems tells us that every linearly independent set of vectors in the vector space $V$ is contained in a basis for $V$. The first Theorem about vector space dimension tells us that every basis for $V$ has $n$ vectors. It is not possible to adjoin extra vectors to the set $u_1, \ldots, u_n$ in order to produce a basis for $V$. The only remaining option is that $u_1, \ldots, u_n$ is already a basis for $V$. In other words, $u_1, \ldots, u_n$ already span all of $V$ and every vector in $V$ is already in $U$.

8. (5 points) Let $v_1, v_2, v_3$ be linearly independent vectors in $\mathbb{R}^3$. Can every vector in $\mathbb{R}^3$ be written in terms of $v_1, v_2, v_3$ in a unique way? If the answer is yes, then PROVE the assertion. If the answer is no, then give a COUNTER EXAMPLE.
YES. The three linearly independent vectors \( v_1, v_2, v_3 \) in the 3-dimensional vector space \( \mathbb{R}^3 \) are automatically a basis for \( \mathbb{R}^3 \). It is always true that if \( v_1, \ldots, v_n \) is a basis for a vector space \( V \), then every vector in \( V \) may be written in terms of the basis \( v_1, \ldots, v_n \) in a unique manner.

9. (5 points) Let \( v_1, v_2, v_3, v_4 \) be vectors in \( \mathbb{R}^4 \). Suppose that \( v_1, v_2, v_3 \) are linearly independent; \( v_1, v_2, v_4 \) are linearly independent; \( v_1, v_3, v_4 \) are linearly independent; and \( v_2, v_3, v_4 \) are linearly independent. Do \( v_1, v_2, v_3, v_4 \) have to be linearly independent? If the answer is yes, then PROVE the assertion. If the answer is no, then give a COUNTER EXAMPLE.

NO. Let

\[
\begin{align*}
  v_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & v_2 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & v_3 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & v_4 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.
\end{align*}
\]

It is clear that \( v_1, v_2, v_3 \) are linearly independent; \( v_1, v_2, v_4 \) are linearly independent; \( v_1, v_3, v_4 \) are linearly independent; \( v_2, v_3, v_4 \) are linearly independent; but \( v_1, v_2, v_3, v_4 \) are linearly dependent.