**Math 544, Exam 2 Solutions, Summer 2004**

Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet; start each computational problem on a new sheet of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

There are 10 problems. Each problem is worth 5 points. The exam is worth a total of 50 points. SHOW your work. CIRCLE your answer. CHECK your answer whenever possible. **No Calculators.**

If I know your e-mail address, I will e-mail your grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail.

I will leave your exam outside my office door by noon tomorrow, you may pick it up any time between then and the next class.

I will post the solutions on my website shortly after the class is finished.

1. **Find the GENERAL solution of the system of linear equations** $Ax = b$.

   Also, list three SPECIFIC solutions. CHECK that the specific solutions satisfy the equations.

   $$
   A = \begin{bmatrix}
   1 & 2 & -1 & 1 & 10 \\
   1 & 2 & -1 & 2 & 16 \\
   2 & 4 & -2 & 3 & 26
   \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}
   x_1 \\
   x_2 \\
   x_3 \\
   x_4 \\
   x_5
   \end{bmatrix}, \quad b = \begin{bmatrix}
   5 \\
   5 \\
   10
   \end{bmatrix}.
   $$

   Apply the row operations $R_2 \leftrightarrow R_2 - R_1$ and $R_3 \leftrightarrow R_3 - 2R_1$ to the matrix

   $\begin{bmatrix}
   1 & 2 & -1 & 1 & 10 & | & 5 \\
   1 & 2 & -1 & 2 & 16 & | & 5 \\
   2 & 4 & -2 & 3 & 26 & | & 10
   \end{bmatrix}$

   to obtain

   $\begin{bmatrix}
   1 & 2 & -1 & 1 & 10 & | & 5 \\
   0 & 0 & 0 & 1 & 6 & | & 0 \\
   0 & 0 & 0 & 1 & 6 & | & 0
   \end{bmatrix}$.

   Apply $R_3 \leftrightarrow R_3 - R_2$ and $R_1 \leftrightarrow R_1 - R_2$ to get

   $\begin{bmatrix}
   1 & 2 & -1 & 0 & 4 & | & 5 \\
   0 & 0 & 0 & 1 & 6 & | & 0 \\
   0 & 0 & 0 & 0 & 0 & | & 0
   \end{bmatrix}$.

   The general solution of the system of equations is

   $$
   \begin{pmatrix}
   x_1 \\
   x_2 \\
   x_3 \\
   x_4 \\
   x_5
   \end{pmatrix} = \begin{pmatrix}
   5 \\
   0 \\
   0 \\
   0 \\
   0
   \end{pmatrix} + x_2 \begin{pmatrix}
   -2 \\
   1 \\
   0 \\
   0 \\
   0
   \end{pmatrix} + x_3 \begin{pmatrix}
   1 \\
   0 \\
   1 \\
   0 \\
   0
   \end{pmatrix} + x_5 \begin{pmatrix}
   -4 \\
   0 \\
   0 \\
   -6 \\
   1
   \end{pmatrix}, \quad x_2, x_3, x_5 \in \mathbb{R}.
   $$
Some specific solutions of this system of equations are

\[
\begin{align*}
v_1 &= \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
v_3 &= \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ -6 \end{bmatrix}, \\
v_4 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\end{align*}
\]

(In \(v_1\), I took \(x_2 = x_3 = x_5 = 0\). In \(v_2\), I took \(x_2 = 1\), \(x_3 = 0\), and \(x_5 = 0\). In \(v_3\), I took \(x_2 = 0\), \(x_3 = 1\), and \(x_5 = 0\). In \(v_4\), I took \(x_2 = 0\), \(x_3 = 0\), and \(x_5 = 1\).) We check

\[
\begin{align*}
Av_1 &= \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 \\ 5 \cdot 1 \\ 5 \cdot 2 \end{bmatrix} = b. \checkmark
\end{align*}
\]

\[
\begin{align*}
Av_2 &= \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 2 \cdot 1 \\ 3 \cdot 1 + 2 \cdot 1 \\ 3 \cdot 2 + 1 \cdot 4 \end{bmatrix} = b. \checkmark
\end{align*}
\]

\[
\begin{align*}
Av_3 &= \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 1 \cdot (-1) \\ 6 \cdot 1 + 1 \cdot (-1) \\ 6 \cdot 2 + 1 \cdot (-2) \end{bmatrix} = b. \checkmark
\end{align*}
\]

\[
\begin{align*}
Av_4 &= \begin{bmatrix} 1 & 2 & -1 & 1 & 10 \\ 1 & 2 & -1 & 2 & 16 \\ 2 & 4 & -2 & 3 & 26 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 - 6 \cdot 1 + 1 \cdot 10 \\ 1 \cdot 1 - 6 \cdot 2 + 1 \cdot 16 \\ 1 \cdot 2 - 6 \cdot 3 + 1 \cdot 26 \end{bmatrix} = b. \checkmark
\end{align*}
\]

2. **Express** \(a = \begin{bmatrix} 11 \\ 16 \end{bmatrix}\) **as a linear combination of** \(v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\) **and** \(v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}\).

We want to find \(c_1\) and \(c_2\) with

\[
(1) \quad c_1 v_1 + c_2 v_2 = a.
\]

This amounts to solving a system of equations. Apply \(R_2 \leftrightarrow R_2 - 2R_1\) to the matrix

\[
\begin{bmatrix} 1 & 3 & 11 \\ 2 & 4 & 16 \end{bmatrix}
\]

to obtain

\[
\begin{bmatrix} 1 & 3 & 11 \\ 0 & -2 & -6 \end{bmatrix}.
\]
Divide row 2 by $-2$ to obtain
\[
\begin{bmatrix}
1 & 3 & 11 \\
0 & 1 & 3
\end{bmatrix}
\]

Apply $R_1 \mapsto R_1 - 3R_2$ to obtain
\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 3
\end{bmatrix}
\]

Thus, the solution of (1) is $c_1 = 2$ and $c_2 = 3$. Observe that
\[
2v_1 + 3v_2 = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 + 9 \\ 4 + 12 \end{bmatrix} = a,
\]
as expected; so the answer is $a = 2v_1 + 3v_2$.

3. Are the vectors $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ linearly independent or linearly dependent? Explain.

We solve the system of equations
\[
(2) \quad c_1 v_1 + c_2 v_2 + c_3 v_3 = 0.
\]

Apply $R_2 \mapsto R_2 - 2R_1$ and $R_3 \mapsto R_3 - 3R_1$ to
\[
\begin{bmatrix}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9
\end{bmatrix}
\]
to obtain
\[
\begin{bmatrix}
1 & 4 & 7 \\
0 & -3 & -6 \\
0 & -6 & -12
\end{bmatrix}
\]

Divide row 2 by $-3$ to see that
\[
\begin{bmatrix}
1 & 4 & 7 \\
0 & 1 & 2 \\
0 & -6 & -12
\end{bmatrix}
\]

Apply $R_3 \mapsto R_3 + 6R_2$ and $R_1 \mapsto R_1 - 4R_2$ to obtain
\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]

The general solution of (2) is $c_1 = c_3$, $c_2 = -2c_3$ and $c_3$ is a free variable. In particular, if we take $c_3 = 1$, then we have $c_1 = 1$, $c_2 = -2$, and $c_3 = 1$. It is indeed true that $1v_1 - 2v_2 + 1v_3 = 0$ because $1 - 8 + 7 = 0$, $2 - 10 + 8 = 0$ and $3 - 12 + 9 = 0$.

The vectors $v_1$, $v_2$, $v_3$ are linearly DEPENDENT.
4. The vectors $v_1$, $v_2$, and $v_3$ are linearly independent. Do the vectors $v_1 - v_2$, $v_2 - v_3$, and $v_3 - v_1$ have to be linearly independent? If yes, then prove the result. If no, then give an example.

\textbf{No.} In fact, the vectors $v_1 - v_2$, $v_2 - v_3$, and $v_3 - v_1$ are ALWAYS linearly DEPENDENT because

$$1(v_1 - v_2) + 1(v_2 - v_3) + 1(v_3 - v_1) = 0$$

for EVERY choice of $v_1$, $v_2$, and $v_3$. If you want a concrete example, the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent; but the vectors

$$w_1 = v_1 - v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad w_2 = v_2 - v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad w_3 = v_3 - v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are linearly dependent since

$$1w_1 + 1w_2 + 1w_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is most likely that the answer to this problem did not just jump into your head. You probably set out to decide if $v_1 - v_2$, $v_2 - v_3$, and $v_3 - v_1$ are linearly independent. So you set out to solve

$$c_1(v_1 - v_2) + c_2(v_2 - v_3) + c_3(v_3 - v_1) = 0. \quad (3)$$

Equation (3) is equivalent to

$$(c_1 - c_3)v_1 + (c_2 - c_1)v_2 + (c_3 - c_2)v_3 = 0. \quad (4)$$

The vectors $v_1$, $v_2$, and $v_3$ are linearly independent; consequently the coefficients which appear in (4) MUST be zero; that is,

$$\begin{cases} c_1 - c_3 = 0 \\ c_2 - c_1 = 0 \\ c_3 - c_2 = 0 \end{cases} \quad (5)$$

We know how to solve (5). Apply $R_2 \leftrightarrow R_2 + R_1$ to

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$
to obtain
\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{bmatrix}.
\]

Apply \( R_3 \mapsto R_3 + R_2 \) to obtain
\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}.
\]

The solution to (5) is \( c_1 = c_3 \), \( c_2 = c_3 \) and \( c_3 \) is a free variable. If we take \( c_3 = 1 \), then \( c_1 = c_2 = c_3 = 1 \) is a solution of (5). Hence \( c_1 = c_2 = c_3 = 1 \) is a solution of (3), and that is where my answer began.

5. **Let** \( A \) **be an** \( n \times n \) **matrix with the property that** \( Ax = b \) **has a unique solution for every vector** \( b \) **in** \( \mathbb{R}^n \). **Does** \( A^T x = b \) **have to have a unique solution for every vector** \( b \) **in** \( \mathbb{R}^n \)? **If yes, then** **prove the result. If no, then give an example.**

YES The hypothesis tells us that every condition of the non-singular matrix theorem holds for the matrix \( A \). In particular, the matrix \( A \) has an inverse. It follows that the matrix \( A^T \) has an inverse. (Indeed, the inverse of \( A^T \) is merely the transpose of \( A^{-1} \), as we saw in class.) Every condition in the non-singular matrix theorem holds for the matrix \( A^T \). In particular, the system of equations \( A^T x = b \) has a unique solution for every vector \( b \) in \( \mathbb{R}^n \).

6. **Define** “linearly independent”. **Use complete sentences.**

The vectors \( v_1, \ldots, v_p \) in \( \mathbb{R}^m \) are **linearly independent** if the ONLY numbers \( c_1, \ldots, c_p \), with \( c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0 \) are \( c_1 = c_2 = \cdots = c_p = 0 \).

7. **Define** “non-singular”. **Use complete sentences.**

The square matrix \( A \) is **non-singular** if the only column vector \( x \) with \( Ax = 0 \) is \( x = 0 \).

8. **State the result about the linear dependence of** \( p \) **vectors in** \( m \)-space. **(I call this the Short Fat Theorem).**

If \( p > m \), then any list of \( p \) vectors from \( \mathbb{R}^m \) is linearly DEPENDENT.

9. **Let** \( W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \bigg| \begin{array}{l} x_1+2x_2+3x_3=0 \\ 2x_1+4x_2+6x_3=0 \\ x_1-7x_2+9x_3=0 \end{array} \right\} \). **Is** \( W \) **a vector space?**

YES The set \( W \) is the null space of the matrix \( A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & -7 & 9 \end{bmatrix} \). We proved in class that the null space of any matrix is a vector space.
10. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \mid 0 \leq x_1 x_2 \right\}$. Is $W$ a vector space? Explain.

NO. The set $W$ is not closed under addition because $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ are in $W$, but the sum $v_1 + v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is not in $W$. 
