1. Define “basis”. Use complete sentences.

2. State any TWO of the four theorems about dimension. Use complete sentences.

3. Give an example of a 4 × 3 matrix A of rank 2. (Recall that the rank of a matrix is the dimension of its column space.)
   (a) For your matrix A, which vectors b have the property that Ax = b has a solution.
   (b) For your matrix A, give an example of a non-zero vector b for which Ax = b DOES have a solution.
   (c) For your matrix A, give an example of a non-zero vector b for which Ax = b does NOT have a solution.

4. Let W be the set of all polynomials f(x) of degree less than or equal to 3 with f(3) = 0. Is W a vector space? If YES, then give a BASIS for W, no proof is needed. If NO, give an EXAMPLE which shows that W is not closed under addition or scalar multiplication.
5. Suppose that $T: \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with

$$
T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.
$$

Find $T \left( \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right)$.

6. Let $W$ be the set of $2 \times 2$ singular matrices. Is $W$ a vector space? If YES, then give a BASIS for $W$, no proof is needed. If NO, give an EXAMPLE which shows that $W$ is not closed under addition or scalar multiplication.


9. Prove that every $2 \times 2$ symmetric matrix has at least one real eigenvalue.

10. Let $v_1, v_2, v_3$ be an orthogonal set of non-zero vectors. Prove that $v_1, v_2, v_3$ are linearly independent.

11. Consider the system of linear equations.

$$
\begin{align*}
4x_1 + ax_2 &= 4 \\
ax_1 + 4x_2 &= 4.
\end{align*}
$$

(a) Which values for $a$ cause the system to have no solution?
(b) Which values for $a$ cause the system to have exactly one solution?
(c) Which values for $a$ cause the system to have an infinite number of solutions?

Explain.
12. Suppose that $A$ and $B$ are $2 \times 2$ matrices with $A$ non-singular. How is the null space of $B$ related to the null space of $AB$, if at all? Prove your answer.

13. Let

$$
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & -1 \\
1 & -1 & 0 \\
1 & 0 & 1 
\end{bmatrix}
$$

and

$$
b = \begin{bmatrix}
1 \\
-1 \\
1 \\
-1
\end{bmatrix}.
$$

Solve $Ax = b$. **Check your answer.** You might want to notice that the columns of $A$ are an orthogonal set.

14. Find an orthogonal basis for the null space of $A = [1 \ 2 \ 3 \ 5]$. **Check your answer.**

15. Let $A = \begin{bmatrix} 14 & 10 \\ -5 & -1 \end{bmatrix}$. Find a matrix $B$ with $B^2 = A$. (I want to see the four entries in the matrix $B$.) **Check your answer.**

16. **Yes or No.** The vectors $v_1$, $v_2$, and $v_3$ are linearly independent. Are the vectors $v_1 + 2v_2 + 3v_3$, $4v_1 + 5v_2 + 6v_3$, and $7v_1 + 8v_2 + 9v_3$ also linearly independent? If yes, give a proof. If no, give an example.

17. Let

$$
A = \begin{bmatrix}
1 & 4 & 5 & 1 & 1 & 1 \\
1 & 4 & 5 & 2 & 3 & 4 \\
2 & 8 & 10 & 3 & 4 & 5 
\end{bmatrix}
$$

and

$$
b = \begin{bmatrix}
3 \\
5 \\
8
\end{bmatrix}.
$$

(a) Find the general solution of $Ax = b$.
(b) Find three particular solutions of $Ax = b$.
(c) Check that your particular solutions work.
(d) Find a basis for the column space of $A$.
(e) Find a basis for the null space of $A$.
(f) Find a basis for the row space of $A$.
(g) Express each column of $A$ as a linear combination of the vectors in your answer to (d).

(h) Express each row of $A$ as a linear combination of the vectors in your answer to (f).