SOLUTIONS to the FINAL Exam, Math 544, Spring, 2003
PRINT Your Name: ________________________________________

There are 20 problems on 12 pages. Each problem is worth 5 points. The exam is worth a total of 100 points. SHOW your work. \[ CIRCLE \] your answer. CHECK your answer whenever possible. \[ \text{No Calculators}. \]

If I know your e-mail address, I will e-mail your course grade to you. If I don’t already know your e-mail address and you want me to know it, send me an e-mail. Otherwise, get your course grade from VIP.

Recall that \( \mathcal{P}_n \) is the vector space of polynomials of degree at most \( n \) with real number coefficients.

Recall that the matrix \( A \) is skew-symmetric if \( A^T = -A \).

1. Suppose that \( T: \mathcal{P}_2 \to \mathcal{P}_4 \) is a linear transformation, where \( T(1) = x^4 \), \( T(x+1) = x^3 - 2x \), and \( T(x^2 + 2x + 1) = x \). Find \( T(x^2 + 5x - 1) \).

Observe that \( x^2 + 5x - 1 = (x^2 + 2x + 1) + 3(x + 1) - 5(1) \); so

\[
T(x^2 + 5x - 1) = T(x^2 + 2x + 1) + 3T(x + 1) - 5T(1) \\
= x + 3(x^3 - 2x) - 5x^4 = \boxed{-5x^4 + 3x^3 - 5x}.
\]

2. Let \( W \) be the subspace of \( \mathcal{P}_4 \) which is defined as follows: the polynomial \( p(x) \) is in \( W \) if and only if \( p(1) + p(-1) = 0 \) and \( p(2) + p(-2) = 0 \). Find the dimension of \( W \). Explain.

Consider the linear transformation \( T: \mathcal{P}_4 \to \mathbb{R}^2 \), which is given by \( T(p(x)) = \begin{bmatrix} p(1) + p(-1) \\ p(2) + p(-2) \end{bmatrix} \). The vector space \( W \) is the null space of \( T \). So the dimension of \( W \) is equal to the dimension of \( \mathcal{P}_4 \) minus the dimension of the image of \( T \). We know that \( \dim \mathcal{P}_4 = 5 \), since \( 1, x, x^2, x^3, x^4 \) is a basis for \( \mathcal{P}_4 \). The image of \( T \) is all of \( \mathbb{R}^2 \) because the image of \( T \) is a subspace of \( \mathbb{R}^2 \) which
contains $T(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $T(x^2) = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$. The vectors $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ span $\mathbb{R}^2$. We conclude that

$$\dim W = 5 - 2 = 3.$$ 

(There are many other ways to reach this answer. The most straightforward thing to do is to calculate a basis for $W$.)

3. Let $W$ be the set of $2 \times 2$ matrices whose trace is zero. Is $W$ a vector space? If YES, then give a basis for $W$, no proof is needed. If NO, give an example which shows that $W$ is not closed under addition or scalar multiplication. Recall that the trace of a square matrix is the sum of its diagonal elements. 

**Yes**, $W$ is a vector space with basis

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

4. Let $W$ be the set of polynomials $p(x)$ in $\mathcal{P}_3$ with $p(0) = 2$. Is $W$ a vector space? If YES, then give a basis for $W$, no proof is needed. If NO, give an example which shows that $W$ is not closed under addition or scalar multiplication. 

**No**, $W$ is not a vector space. The polynomial $p(x) = 2$ is in $W$ but the polynomial $3p(x)$, which is the constant polynomial 6, is not in $W$.

5. Let $W$ be the set of $2 \times 2$ matrices whose determinant is zero. Is $W$ a vector space? If YES, then give a basis for $W$, no proof is needed. If NO, give an example which shows that $W$ is not closed under addition or scalar multiplication.
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 \end{bmatrix}
\]
are in \( W \), but their sum, which is \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), is not in \( W \).

6. Let \( W \) be the set of polynomials \( p(x) \) in \( \mathcal{P}_3 \) with \( p(2) = 0 \). Is \( W \) a vector space? If \( \text{YES} \), then give a basis for \( W \), no proof is needed. If \( \text{NO} \), give an example which shows that \( W \) is not closed under addition or scalar multiplication.

\boxed{\text{Yes}} \), \( W \) is a vector space with basis

\[ x - 2, \ (x - 2)^2, \ (x - 2)^3. \]

7. Find \( \lim_{n \to \infty} A^n \), where \( A = \begin{bmatrix} 2 & 3 \\ -1 & -1 \frac{1}{2} \end{bmatrix} \).

This problem would be easy if \( A \) were a diagonal matrix. Lets diagonalize \( A \). The eigenvalues of \( A \) satisfy

\[ 0 = \det(A - \lambda I) = (2 - \lambda)(-1 - \lambda) + \frac{3}{2} = \lambda^2 - \frac{3}{2} \lambda + \frac{1}{2} \]

\[ = (\lambda - 1)(\lambda - \frac{1}{2}). \]

The eigenvalues of \( A \) are \( \lambda = 1 \) and \( \lambda = \frac{1}{2} \). The eigenvectors which belong to \( \lambda = 1 \) are the nullspace of \( A - I = \begin{bmatrix} 1 & 3 \\ -1 & -1 \frac{3}{2} \end{bmatrix} \).

Replace row 2 by row 2 plus row 1 to get \( \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \). The eigenspace which belongs to \( \lambda = 1 \) is \( x_1 = -\frac{3}{2} x_2 \) and \( x_2 \) can be anything.

The vector \( v_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \) is a basis for the eigenspace which belongs to \( \lambda = 1 \). By the way

\[ Av_1 = \begin{bmatrix} 2 & 3 \\ -1 & -1 \frac{1}{2} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 + 3 \\ 3 - 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = v_1, \]
as expected. The eigenspace which belongs to \( \lambda = \frac{1}{2} \) is the null space of \( A - \frac{1}{2} = \begin{bmatrix} 3/2 & 3/2 \\ -1 & -1 \end{bmatrix} \). Exchange the two rows:
\[
\begin{bmatrix}
-1 & -1 \\
3/2 & 3/2
\end{bmatrix}
\]. Replace row 2 by row 2 plus \( \frac{3}{2} \) row 1:
\[
\begin{bmatrix}
-1 & -1 \\
0 & 0
\end{bmatrix}
\].
Multiply row 1 by \(-1\):
\[
\begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
\]. The eigenspace which belongs to \( \lambda = \frac{1}{2} \) is \( x_1 = -x_2 \) and \( x_2 \) can be anything. The vector \( v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) is a basis for the eigenspace which belongs to \( \lambda = \frac{1}{2} \).
By the way
\[
Av_2 = \begin{bmatrix}
2 & \frac{3}{2} \\
-1 & -\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
-1 \\
1
\end{bmatrix}
= \begin{bmatrix}
-2 + \frac{3}{2} \\
1 - \frac{1}{2}
\end{bmatrix}
= \begin{bmatrix}
1/2 \\
1/2
\end{bmatrix}
= \begin{bmatrix}
1/2 \\
1
\end{bmatrix}
= \frac{1}{2}v_2 ,
\]
as expected. Now we know that
\[
A \begin{bmatrix}
-3 & -1 \\
2 & 1
\end{bmatrix}
= \begin{bmatrix}
-3 & -1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1/2
\end{bmatrix} .
\]
Let
\[
D = \begin{bmatrix}
1 & 0 \\
0 & 1/2
\end{bmatrix} , \quad \text{and} \quad S = \begin{bmatrix}
-3 & -1 \\
2 & 1
\end{bmatrix} .
\]
We calculate that \( S^{-1} = \begin{bmatrix}
-1 & -1 \\
2 & 3
\end{bmatrix} . \) We saw that \( AS = SD \).
It follows that \( A = SDS^{-1} \) and
\[
\lim_{n \to \infty} A^n S = \lim_{n \to \infty} D^n S^{-1} = \begin{bmatrix}
-3 & -1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & -1 \\
2 & 3
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-3 & -1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & -1 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
3 & 3 \\
-2 & -2
\end{bmatrix}
\]
The function $T$ from the vector space $V$ to the vector space $W$ is a **linear transformation** if $T(v_1 + v_2) = T(v_1) + T(v_2)$ and $T(rv_1) = rT(v_1)$ for all $v_1$ and $v_2$ in $V$ and all $r$ in $\mathbb{R}$.

9. **Define “eigenvector”.** Use complete sentences.
Let $A$ be a square matrix. The vector $v$ is an **eigenvector** of $A$ belonging to the eigenvalue $\lambda$ provided $Av = \lambda v$ and $Aw = \lambda w$ for some non-zero vector $w$.

10. **Define “linearly independent”.** Use complete sentences.
The vectors $v_1, \ldots, v_p$ in the vector space $V$ are **linearly independent** if the only numbers $c_1, \ldots, c_p$ with $\sum_{i=1}^{p} c_i v_i = 0$ are $c_1 = \cdots = c_p = 0$.

11. **Define “non-singular”.** Use complete sentences. The square matrix $A$ is **non-singular** if the only column vector $x$ with $Ax = 0$ is $x = 0$.

12. **Define “null space”.** Use complete sentences.
The **null space** of the matrix $A$ is the set of all vectors $x$ with $Ax = 0$.

13. **True or False.** (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.) If $A$ is a $2 \times 2$ skew-symmetric matrix, then $A$ has at least one real eigenvalue.

   **False** The eigenvectors of \[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \] are $\pm i$.

14. **True or False.** (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.) Every $4 \times 4$ skew-symmetric matrix is singular.
The matrix
\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]
is non-singular.

15. **True or False.** (If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE.) If \( v_1, v_2, v_3 \) are linearly independent vectors in the vector space \( V \) and \( T: V \to W \) is a linear transformation of vector spaces, then \( T(v_1), T(v_2), T(v_3) \) are linearly independent vectors in the vector space \( W \).

**False** Consider the linear transformation \( T: \mathbb{R}^3 \to \mathbb{R} \) which is multiplication by \( [0 \ 0 \ 0] \). Let
\[
v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

We see that \( v_1, v_2, \) and \( v_3 \) are linearly independent in \( \mathbb{R}^3 \), but \( T(v_1), T(v_2), T(v_3) \) are linearly dependent in \( \mathbb{R} \).

16. **Find the general solution of the system of linear equations** \( Ax = b \). **If the system of equations has more than one solution, then list three SPECIFIC solutions. CHECK that the specific solutions satisfy the equations.**

\[
A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 6 \\ 9 \\ 10 \end{bmatrix}
\]

The same matrix \( A \) appears in problems 16, 17, and 18.
I will do the arithmetic for 16, 17, and 18 all at the same time. Consider the augmented matrix

\[
\begin{bmatrix}
1 & 2 & 1 & 2 & | & 6 & 6 \\
2 & 4 & 1 & 2 & | & 9 & 9 \\
1 & 2 & 2 & 4 & | & 10 & 9
\end{bmatrix}
\]

Replace row 2 by row 2 minus 2 times row 1. Replace row 3 by row 3 minus row 1.

\[
\begin{bmatrix}
1 & 2 & 1 & 2 & | & 6 & 6 \\
0 & 0 & -1 & -2 & | & -3 & -3 \\
0 & 0 & 1 & 2 & | & 4 & 3
\end{bmatrix}
\]

Replace row 1 by row 1 plus row 2. Replace row 3 by row 3 plus row 2

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & | & 3 & 3 \\
0 & 0 & -1 & -2 & | & -3 & -3 \\
0 & 0 & 0 & 0 & | & 1 & 0
\end{bmatrix}
\]

Multiply row 2 by minus 1.

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & | & 3 & 3 \\
0 & 0 & 1 & 2 & | & 3 & 3 \\
0 & 0 & 0 & 0 & | & 1 & 0
\end{bmatrix}
\]

Ignore the column on the far right in problem 16. The bottom row of the augmented matrix

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & | & 3 \\
0 & 0 & 1 & 2 & | & 3 \\
0 & 0 & 0 & 0 & | & 1
\end{bmatrix}
\]
tells us that 0 = 1. This is impossible, so the system of equations has \text{No Solution.}

17. Find the general solution of the system of linear equations \( Ax = b \). If the system of equations has
more than one solution, then list three SPECIFIC solutions. CHECK that the specific solutions satisfy the equations.

\[
A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 6 \\ 9 \\ 9 \end{bmatrix}.
\]

The same matrix \( A \) appears in problems 16, 17, and 18.
Start with the augmented matrix (*) from problem 16. Ignore column five. The resulting matrix is

\[
\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

The general solution set is

\[
\begin{align*}
x_1 &= 3 - 2x_2 \\
x_2 &= x_2 \\
x_3 &= 3 - 2x_4 \\
x_4 &= x_4
\end{align*}
\]

Three specific solutions are:

\[
v_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}.
\]

We check

\[
Av_1 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 9 \end{bmatrix} = b
\]
$$A v_2 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 9 \end{bmatrix} = b\sqrt{\phantom{1}}$$

$$A v_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 9 \end{bmatrix} = b\sqrt{\phantom{1}}$$

18. Find bases for the row space, column space, and null space of

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}$$

The same matrix $A$ appears in problems 16, 17, and 18.

We put $A$ into row reduced echelon form in problem 16. We may read this form from (*), simply ignore columns 5 and 6:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vectors

$$\begin{bmatrix} 1 & 2 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}$$

are a basis for the row space of $A$. The vectors

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are a basis for the column space of $A$. The vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$
are a basis for the null space of $A$.

19. Find the general solution of the system of linear equations $Ax = b$. If the system of equations has more than one solution, then list three SPECIFIC solutions. CHECK that the specific solutions satisfy the equations.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 1 & 3 & 6 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -6 \\ -16 \\ -13 \end{bmatrix}.$$ 

Consider

$$\begin{bmatrix} 1 & 2 & 3 & | & -6 \\ 1 & 3 & 7 & | & -16 \\ 1 & 3 & 6 & | & -13 \end{bmatrix}$$

Replace row 2 with row 2 minus row 1. Replace row 3 with row 3 minus row 1:

$$\begin{bmatrix} 1 & 2 & 3 & | & -6 \\ 0 & 1 & 4 & | & -10 \\ 0 & 1 & 3 & | & -7 \end{bmatrix}$$

Replace row 1 with row 1 minus 2 row 2. Replace row 3 with row 3 minus row 2:

$$\begin{bmatrix} 1 & 0 & -5 & | & 14 \\ 0 & 1 & 4 & | & -10 \\ 0 & 0 & -1 & | & 3 \end{bmatrix}$$

Replace row 2 by row 2 plus 4 row 3. Replace row 1 by row 1 minus 5 row 3:

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & -1 & | & 3 \end{bmatrix}$$

Multiply row 3 by $-1$.

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -3 \end{bmatrix}$$
The solution is \( v = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \). We check this:

\[
Av = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 + 4 - 9 \\ -1 + 6 - 21 \\ -1 + 6 - 18 \end{bmatrix} = \begin{bmatrix} -6 \\ -16 \\ -13 \end{bmatrix} = b\sqrt{3}.
\]

20. **Find an orthogonal basis for the null space of** \( A = \begin{bmatrix} 1 & 2 & 3 & 5 \end{bmatrix} \).

One basis for the null space of \( A \) is

\[
v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\]

We apply the Gram-Schmidt orthogonalization process to this basis. Let \( u_1 = v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \). Let

\[
u_2' = v_2 - \frac{u_1^T v_2}{u_1^T u_1} u_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{6}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.
\]

Let

\[
u_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \\ 0 \end{bmatrix}.
\]

(Notice that \( Au_2 = 0 \) and \( u_1^T u_2 = 0 \).) Let

\[
u_3' = v_3 - \frac{u_1^T v_3}{u_1^T u_1} u_1 - \frac{u_2^T v_3}{u_2^T u_2} u_2 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{15}{70} \begin{bmatrix} -3 \\ 6 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \frac{3}{14} \\ 0 \\ 0 \end{bmatrix}.
\]
\[
\begin{bmatrix}
-1 \\
-2 \\
0 \\
1
\end{bmatrix} + \frac{1}{14} \begin{bmatrix}
9 \\
18 \\
-15 \\
0
\end{bmatrix} = \frac{1}{14} \begin{bmatrix}
-5 \\
-10 \\
-15 \\
14
\end{bmatrix}
\]

Let
\[
u_3 = \begin{bmatrix}
-5 \\
-10 \\
-15 \\
14
\end{bmatrix}.
\]

It is easy to check that
\[
u_1 = \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}, \quad u_2 = \begin{bmatrix}
-3 \\
-6 \\
5 \\
0
\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}
-5 \\
-10 \\
-15 \\
14
\end{bmatrix}
\]

is an orthogonal basis for the null space of \( A \).