1. **Find the GENERAL solution of the system of linear equations** \( Ax = b \). Also, list three SPECIFIC solutions, if possible. **CHECK** that the specific solutions satisfy the equations.

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 & 1 & 6 \\
1 & 2 & 3 & 4 & 2 & 12 \\
2 & 4 & 6 & 8 & 3 & 18
\end{bmatrix}, \quad x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix}, \quad b = \begin{bmatrix}
3 \\
5 \\
8
\end{bmatrix}.
\]

Start with the augmented matrix

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 1 & 6 & 3 \\
1 & 2 & 3 & 4 & 2 & 12 & 5 \\
2 & 4 & 6 & 8 & 3 & 18 & 8
\end{bmatrix}.
\]

Replace row 2 with row 2 minus row 1.
Replace row 3 with row 3 minus 2 times row 1.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 1 & 6 \\
0 & 0 & 0 & 0 & 1 & 6 \\
0 & 0 & 0 & 0 & 1 & 6
\end{bmatrix}
\begin{bmatrix}
3 \\
2 \\
2
\end{bmatrix}
\]

Replace row 1 with row 1 minus row 2.
Replace row 3 with row 3 minus row 2.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix}
\]

The general solution set is

\[
\begin{align*}
x_1 &= 1 -2x_2 -3x_3 -4x_4 \\
x_2 &= x_2 \\
x_3 &= x_3 \\
x_4 &= x_4 \\
x_5 &= 2 -6x_6 \\
x_6 &= x_6
\end{align*}
\]

where \(x_2, x_3, x_4, \text{ and } x_6\) are arbitrary.

Some particular solutions are

\[
v_1 = v_2 = \begin{bmatrix}
-1 \\
1 \\
0 \\
0 \\
2 \\
0
\end{bmatrix},
\quad v_3 = \begin{bmatrix}
-2 \\
0 \\
1 \\
0 \\
2 \\
0
\end{bmatrix},
\quad v_4 = \begin{bmatrix}
-3 \\
0 \\
0 \\
1 \\
2 \\
0
\end{bmatrix},
\quad v_5 = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
-4 \\
1
\end{bmatrix}
\]

These vectors were obtained by setting \(x_2 = x_3 = x_4 = x_6 = 0\) (for \(v_1\)); \(x_2 = 1, x_3 = x_4 = x_6 = 0\) (for \(v_2\)); \(x_2 = 0, x_3 = 1, x_4 = x_6 = 0\) (for \(v_3\)); \(x_2 = x_3 = 0, x_4 = 1, x_6 = 0\) (for \(v_4\)); and \(x_2 = x_3 = x_4 = 0, x_6 = 1\) (for \(v_5\)). We check these specific solutions:
\[ Av_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 1+5 \\ 2+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark \]

\[ Av_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+2+2 \\ -1+2+4 \\ -2+4+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark \]

\[ Av_3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2+3+2 \\ -2+3+4 \\ -4+6+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark \]

\[ Av_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3+4+2 \\ -3+4+4 \\ -6+8+6 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark \]

\[ Av_5 = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 2 & 12 \\ 2 & 4 & 6 & 8 & 3 & 18 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-4+6 \\ 1-8+12 \\ 2-12+18 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \checkmark \]
2. Consider the system of linear equations.

\[
\begin{align*}
x_1 + (a - 1)x_2 &= 4 \\
a x_1 + 6x_2 &= 12.
\end{align*}
\]

(a) Which values for \( a \) cause the system to have no solution?
(b) Which values for \( a \) cause the system to have exactly one solution?
(c) Which values for \( a \) cause the system to have an infinite number of solutions?

Explain thoroughly.

Consider the augmented matrix

\[
\begin{bmatrix}
1 & a - 1 & 4 \\
a & 6 & 12
\end{bmatrix}.
\]

Replace row 2 by row 2 minus \( a \) times row 1 to get:

\[
\begin{bmatrix}
1 & a - 1 & 4 \\
0 & 6 - a(a - 1) & 12 - 4a
\end{bmatrix}.
\]

Multiply row 2 by \(-1\) to get

\[
\begin{bmatrix}
1 & a - 1 & 4 \\
0 & a^2 - a - 6 & 4a - 12
\end{bmatrix}.
\]

which is the same as

\[
\begin{bmatrix}
1 & a - 1 & 4 \\
0 & (a - 3)(a + 2) & 4(a - 3)
\end{bmatrix}.
\]

We see that if \( a \) is any number other than 3 or \(-2\), then the system of equations has a unique solution. (In this case the bottom row tells the value of \( x_2 \) and the top row tells the value of \( x_1 \).) If \( a = 3 \), then the bottom row is completely zero and the solution set is the line described by the top equation. If \( a = -2 \), then the system of equations has no solution because 0 is never equal to \(-20\). To summarize:

| (a) If \( a = -2 \), then the system of equations has no solution. |
| (b) If \( a \neq -2 \) and \( a \neq 3 \), then the system of equations has exactly one solution. |
| (b) If \( a = 3 \), then the system of equations has an infinite number of solutions. |
3. Are the vectors

\[
\begin{align*}
v_1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \\
v_3 &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}
\end{align*}
\]

linearly independent? Explain thoroughly.

These vectors are DEPENDENT because \(-2v_1 + v_2 + v_3 = 0\). We have exhibited a non-trivial linear combination of the vectors which equals the zero vector.

4. Suppose \(v_1, v_2\) and \(v_3\) are vectors in \(\mathbb{R}^3\) with \(v_1, v_2\) linearly independent, \(v_1, v_3\) linearly independent, and \(v_2, v_3\) linearly independent. Do the vectors \(v_1, v_2, v_3\) have to be linearly independent? If yes, give a proof. If no, give an example.

NO. The vectors of problem 3 give an example. The vectors \(v_1\) and \(v_2\) are independent (since neither vector is a multiple of the other). The vectors \(v_1\) and \(v_3\) are independent (since neither vector is a multiple of the other). The vectors \(v_2\) and \(v_3\) are independent (since neither vector is a multiple of the other). Yet the vectors \(v_1, v_2, v_3\) are dependent because \(-2v_1 + v_2 + v_3 = 0\).

5. How many solutions does a homogeneous system of 3 linear equations in 4 unknowns have? Justify your answer very thoroughly.

This system of equations MUST have an infinite number of solutions. It is guaranteed at least one solution because it is a homogeneous system of equations. Furthermore, there are more variables than equations so some column in the reduced matrix does not contain a leading one. This column corresponds to a variable which is free to take on any value.

6. How many solutions does a homogeneous system of 4 linear equations in 3 unknowns have? Justify your answer very thoroughly.

This system of equations MIGHT have a unique solution or MIGHT have an infinite number of solutions. The system of equations is homogeneous, so the system has at least one solution. At any rate, the reduced matrix will have at most 3 leading ones. If there are exactly 3 leading ones, then the system of equations has a unique solution. If there are fewer than 3 leading ones (In other words, if there
were “really” only two equations and the other two equations can be expressed as linear combinations of the first two equations.), then the system of equations would have an infinite number of solutions.

7. **Recall that the matrix** \( A \) **is symmetric if** \( A^T = A \). **Let** \( A \) **and** \( B \) **be** \( 2 \times 2 \) **symmetric matrices. Give an example to show that** \( AB \) **does not have to be a symmetric matrix.**

Let \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). We see that \( A \) and \( B \) are each symmetric, but the product
\[
AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\]
is not symmetric.

8. **Give a condition (**) so that if** \( A \) **and** \( B \) **are** \( 2 \times 2 \) **symmetric matrices which satisfy (**) , then** \( AB \) **also is a symmetric matrix.**

If \( AB = BA \), then \( AB \) is also a symmetric matrix because
\[
(AB)^T = B^T A^T = BA = AB.
\]
The first equality holds for all matrices. The send equality holds because \( B \) and \( A \) are symmetric. The last equality is our new hypothesis.

9. **List four different** \( 2 \times 2 \) **matrices** \( X \) **which satisfy** \( X^2 - 2X = 0 \).

The matrices
\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & b \\ \frac{1}{b} & 1 \end{bmatrix}
\]
all work for any number \( b \neq 0 \).
10. **Find a matrix** $B$ **with** $AB = C$ **for** $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ **and** $C = \begin{bmatrix} 2 & 6 \\ 3 & 6 \end{bmatrix}$.

Let $B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$. We solve $A \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and we solve $A \begin{bmatrix} b_{1,2} \\ b_{2,2} \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}$.

We solve the two systems of equations simultaneously. That is, we consider the augmented matrix

$$
\begin{bmatrix}
1 & 3 & | & 2 & 6 \\
1 & 4 & | & 3 & 6
\end{bmatrix}.
$$

When we put the matrix in Reduced Row Echelon Form, the left most column of the augmentation will tell us about $\begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix}$ and the right most column of the augmentation will tell us about $\begin{bmatrix} b_{1,2} \\ b_{2,2} \end{bmatrix}$. Replace row 2 by row 2 minus row 1 to get:

$$
\begin{bmatrix}
1 & 3 & | & 2 & 6 \\
0 & 1 & | & 1 & 0
\end{bmatrix}.
$$

Replace row 1 by row 1 minus 3 times row 2 to get:

$$
\begin{bmatrix}
1 & 0 & | & -1 & 6 \\
0 & 1 & | & 1 & 0
\end{bmatrix}.
$$

We conclude that

$$B = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}.
$$

This is correct because

$$AB = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 6 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1+3 & 6+0 \\ -1+4 & 6+0 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 3 & 6 \end{bmatrix} = C. \checkmark$$