Math 544, Exam 2, Fall 2005

Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you. There are 10 problems. Each problem is worth 5 points. SHOW your work. CIRCLE your answer. CHECK your answer whenever possible. No Calculators.

If I know your e-mail address, I will e-mail your grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail.

I will post the solutions on my website shortly after the exam is finished.

1. Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.

$$A = \begin{bmatrix} 1 & 2 & 2 & 10 \\ 1 & 2 & 3 & 13 \\ 2 & 4 & 5 & 23 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}.$$

We apply Gaussian Elimination to the augmented matrix

$$\begin{bmatrix} 1 & 2 & 2 & 10 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ 2 & 4 & 5 & 23 & -3 \end{bmatrix}.$$

Replace $R2 \leftrightarrow R2 - R1$ and $R3 \leftrightarrow R3 - 2R1$ to get

$$\begin{bmatrix} 1 & 2 & 2 & 10 & -1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & -1 \end{bmatrix}.$$

Replace $R1 \leftrightarrow R1 - 2R2$ and $R3 \leftrightarrow R3 - R2$ to get

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
The general solution is
\[
\begin{align*}
\begin{bmatrix}
    x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} & \in \mathbb{R}^4
\end{align*}
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & -2 & -4 \\
    0 & -1 & 0 & -3 \\
    0 & -1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3 \\
x_4
\end{bmatrix}
\text{ for some numbers } x_2 \text{ and } x_4
\end{align*}
\]

Specific Solution 1. Take \( x_2 = x_4 = 0 \). We see that \( v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \) is a solution of \( Ax = b \) because
\[
Av_1 = A = \begin{bmatrix}
1 & 2 & 2 & 10 \\
1 & 2 & 3 & 13 \\
2 & 4 & 5 & 23
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
-1 \\
0
\end{bmatrix} = \begin{bmatrix}
1 - 2 \\
1 - 3 \\
2 - 5
\end{bmatrix} = \begin{bmatrix}
-1 \\
-2 \\
-3
\end{bmatrix} = b. \checkmark
\]

Specific Solution 2. Take \( x_2 = 1 \) and \( x_4 = 0 \). We see that \( v_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \) is a solution of \( Ax = b \) because
\[
Av_2 = A = \begin{bmatrix}
1 & 2 & 2 & 10 \\
1 & 2 & 3 & 13 \\
2 & 4 & 5 & 23
\end{bmatrix}
\begin{bmatrix}
-1 \\
1 \\
-1 \\
0
\end{bmatrix} = \begin{bmatrix}
-1 + 2 - 2 \\
-1 + 2 - 3 \\
-2 + 4 - 5
\end{bmatrix} = \begin{bmatrix}
-1 \\
-2 \\
-3
\end{bmatrix} = b. \checkmark
\]

Specific Solution 3. Take \( x_2 = 0 \) and \( x_4 = 1 \). We see that \( v_3 = \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \) is a solution of \( Ax = b \) because
\[
Av_3 = A = \begin{bmatrix}
1 & 2 & 2 & 10 \\
1 & 2 & 3 & 13 \\
2 & 4 & 5 & 23
\end{bmatrix}
\begin{bmatrix}
-3 \\
0 \\
-4 \\
1
\end{bmatrix} = \begin{bmatrix}
-3 - 8 + 10 \\
-3 - 12 + 13 \\
-6 - 20 + 23
\end{bmatrix} = \begin{bmatrix}
-1 \\
-2 \\
-3
\end{bmatrix} = b. \checkmark
\]
2. Let $U$ and $V$ be subspaces of $\mathbb{R}^n$. Does the union $U \cup V$ have to be a subspace of $\mathbb{R}^n$? If yes, prove your answer. If no, give a counterexample.

**NO.** Let $n = 2$, $U = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$, and $V = \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} \mid b \in \mathbb{R} \right\}$. We see that $U$ and $V$ are subspaces of $\mathbb{R}^2$; but the union $U \cup V$ is not a subspace of $\mathbb{R}^2$ because this union is not closed under addition. In particular, $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in $U$ (hence also in $U \cup V$) and $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in $V$ (hence also in $U \cup V$); but the sum $u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in either $U$ or $V$; so $u + v \notin U \cup V$.

3. Let $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid \begin{array}{l} x_1 + 3x_2 + 4x_3 = 0 \\ 2x_1 + 9x_2 + 5x_3 = 0 \\ 5x_1 + 14x_2 + 41x_3 = 0 \\ -x_1 + 32x_2 + 12x_3 = 0 \end{array} \right\}$. Is $V$ a vector space?

**Yes.** The set $V$ is the null space of the matrix
\[
\begin{bmatrix}
1 & 3 & 4 \\
2 & 9 & 5 \\
5 & 14 & 41 \\
-1 & 32 & 12
\end{bmatrix}.
\]
We proved that the null space of every matrix is a vector space.

4. Let $V = \left\{ \begin{bmatrix} x_1 + 3x_2 + 4x_3 \\ 2x_1 + 9x_2 + 5x_3 \\ 5x_1 + 14x_2 + 41x_3 \\ -x_1 + 32x_2 + 12x_3 \end{bmatrix} \in \mathbb{R}^4 \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$. Is $V$ a vector space? Explain thoroughly.

**Yes.** The set $V$ is the column space of the matrix
\[
\begin{bmatrix}
1 & 3 & 4 \\
2 & 9 & 5 \\
5 & 14 & 41 \\
-1 & 32 & 12
\end{bmatrix}.
\]
We proved that the column space of every matrix is a vector space.
5. Let $V = \left\{ \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right] \in \mathbb{R}^2 \mid x_1 x_2 = 0 \right\}$. Is $V$ a vector space? Explain thoroughly.

\[ \text{NO.} \] The set $V$ is not closed under addition. We see that $u = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]$ and $v = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$ are in $V$; but $u + v = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$ is not $V$.

6. Define “null space”. Use complete sentences. Include everything that is necessary, but nothing more.

The null space of the matrix $A$ is the set of all column vectors $x$ with $Ax = 0$.

7. (a) Define “non-singular”. Use complete sentences. Include everything that is necessary, but nothing more.

The square matrix $A$ is non-singular if the only column vector $x$ with $Ax = 0$ is $x = 0$.

(b) Let $A$ be an $n \times n$ matrix. List three statements that are equivalent to the statement “$A$ is non-singular”.

1. The columns of $A$ are linearly independent.
2. For every vector $b$ in $\mathbb{R}^n$, the system of equations $Ax = b$ has a unique solution.
3. The matrix $A$ is invertible.

8. Let $A$ and $B$ be $2 \times 2$ matrices with $A$ not equal to the zero matrix and $A^2 = AB$. Does $A$ have to equal $B$? If yes, prove your answer. If no, give a counterexample.

\[ \text{NO.} \] Let $A = \left[ \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right]$ and $B = \left[ \begin{array}{cc} 75 & 82 \\ 0 & 0 \end{array} \right]$. We see that $A$ is not the zero matrix and $A$ does not equal $B$; but, $A^2$ and $AB$ are both equal to $\left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]$.

9. Let $A$ and $B$ be $n \times n$ matrices. At least one of the following statements is always true. Pick a true statement and prove it.

(a) The column space of $A$ is a subset of the column space of $AB$.
(b) The column space of $B$ is a subset of the column space of $AB$. 
(c) The column space of $AB$ is a subset of the column space of $A$.
(d) The column space of $AB$ is a subset of the column space of $B$.

Statement (c) is true. If $v$ is in the column space of $AB$, then $v = ABw$ for some vector $w$. Thus, $v = A(Bw)$ for the vector $Bw$ and $v$ is in the column space of $A$.

10. Let $A$ and $B$ be $n \times n$ matrices. At least one of the following statements is always true. Pick a true statement and prove it.
(a) The null space of $A$ is a subset of the null space of $AB$.
(b) The null space of $B$ is a subset of the null space of $AB$.
(c) The null space of $AB$ is a subset of the null space of $A$.
(d) The null space of $AB$ is a subset of the null space of $B$.

Statement (b) is true. If $v$ is in the null space of $B$, then $Bv = 0$. Multiply both sides of the equation on the left by $A$ to see that $ABv = 0$. We now see that $v$ is in the null space of $AB$. 