MATH 544, 1997, FINAL EXAM

PRINT Your Name: ________________________________

There are 18 problems on 7 pages. Problem 1 is worth 14 points. Each of the other problems is worth 8 points. **SHOW your work. CIRCLE your answer. CHECK your answer whenever possible. NO CALCULATORS.**

1. Let \( A \) be an \( n \times n \) matrix. List 8 statements that are equivalent to the statement “\( A \) is nonsingular”.

2. Define “linear transformation”.

3. Define “null space”.

4. Define “span”.

5. Let \( V \) be the vector space of polynomials \( f(x) \) of degree at most three with \( f(1) = 0 \). Record a basis for \( V \). No justification is needed.

   Let

   \[
   A = \begin{bmatrix}
   1 & 2 & 2 & 6 & 2 & 8 \\
   1 & 2 & 3 & 9 & 2 & 8 \\
   1 & 2 & 3 & 9 & 3 & 12 \\
   2 & 4 & 5 & 15 & 5 & 20 \\
   \end{bmatrix}
   \quad \text{and} \quad
   b = \begin{bmatrix}
   3 \\
   2 \\
   4 \\
   7 \\
   \end{bmatrix}.
   \]

6. Find a basis for the row space of \( A \).

7. Find a basis for the column space of \( A \).

8. Find a basis for the null space of \( A \).

9. Solve \( Ax = b \).

   Let

   \[
   A = \begin{bmatrix}
   5 & 2 & 3 & 7 \\
   3 & 2 & 5 & 7 \\
   \end{bmatrix}.
   \]

10. Find an invertible matrix \( S \) and a diagonal matrix \( D \) with \( S^{-1}AS = D \).

11. Find a matrix \( B \) with \( B^2 = A \).

12. Let \( A \) be a symmetric matrix and let \( u \) and \( v \) be eigenvectors of \( A \) which belong to different eigenvalues. **PROVE** that \( u^T v = 0 \).

13. True or False. If the statement is true, then **PROVE** the statement. If the statement is false, then give a COUNTEREXAMPLE. If \( A \) and \( B \) are \( 2 \times 2 \) matrices with \( A \) non-singular, then the column space of \( AB \) is equal to the column space of \( B \).
14. True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If $A$ and $B$ are $2 \times 2$ symmetric matrices, then $AB$ is a symmetric matrix.

15. True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If $A$ and $B$ are $2 \times 2$ nonsingular matrices, then $AB$ is a nonsingular matrix.

16. True or False. If the statement is true, then PROVE the statement. If the statement is false, then give a COUNTEREXAMPLE. If $A$ and $B$ are $2 \times 2$ nonsingular matrices, then $A + B$ is a nonsingular matrix.

17. Find an orthogonal set which is a basis for the null space of $\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}$. 