Math 174 Fall 2003 Final Exam Solutions

PRINT Your Name:________________________________

There are 25 problems on 7 pages. Each problem is worth 4 points. The exam is worth 100 points. $\boxed{CIRCLE}$ your answers. **No Calculators.**

WHEN YOU DO SOMETHING CLEVER, EXPLAIN YOUR WORK.

If I know your e-mail address, I will e-mail your course grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your course grade from VIP.

I will post the solutions on my website later today.

You may leave the binomial coefficient $\binom{n}{r}$ in any of your answers.

1. **Consider the relation "$\le$" on the set of real numbers. Is this relation reflexive, symmetric, transitive? Explain.**

The relation is reflexive since $a \le a$ for all a. The relation is NOT symmetric since $1 \le 2$, but $2 \not\le 1$. The relation is transitive since if $a \le b$ and $b \le c$, then $a \le c$.

2. **If a and b are integers, then $a \equiv b \mod 5$ if and only if $5|(a-b)$. Describe the equivalance classes that correspond to this equivalence relation.**

There are 5 equivalenance classes:

$$\{\dots,-10,-5,0,5,10,\dots\},\quad \{\dots,-9,-4,1,6,11,\dots\},\quad \{\dots,-8,-3,2,7,12,\dots\}$$

$$\{\dots,-7,-2,3,8,13,\dots\} \quad\text{and}\quad \{\dots,-6,-1,4,9,14,\dots\}.$$

3. **Suppose $n \equiv 1 \mod 5$. Give a formula for $\lfloor \frac{n}{5} \rfloor$ which does not involve $\lfloor\ \rfloor$.**

If $n \equiv 1 \mod 5$, then $\lfloor \frac{n}{5} \rfloor = \frac{n-1}{5}$.

4. **Solve the recurrence relation $a_0 = 1000$, $a_n = (1.05)a_{n-1} + 100$ for $1 \le n$.**

Observe that

$$a_1 = (1.05)1000 + 100, \quad a_2 = (1.05)^2 1000 + (1.05)100 + 100,$$

$$a_3 = (1.05)^3 1000 + (1.05)^2 100 + (1.05)100 + 100.$$

Eventually, we see that

$$a_n = (1.05)^n 1000 + 100((1.05)^{n-1} + (1.05)^{n-2} + \dots + (1.05)^1 + 1)$$

$$= \boxed{(1.05)^n 1000 + 100\frac{(1.05)^n - 1}{(1.05) - 1}.}$$

5. **Consider the Tower of Hanoi problem. There are three towers in a row: tower A, tower B, and tower C. There are n disks of different sizes stacked on tower A. One must move all n disks to tower C. One may NEVER place a bigger disk on top of a smaller disk. In the present problem, one may move a disk only to an ADJACENT tower. Let a_n be the minimum number of moves needed to transfer a stack of n disks from tower A to tower C. Find a_1, a_2, a_3. Find a recurrence relation for $a_1, a_2, a_3, \dots$.**

It is clear that $\boxed{a_1 = 2}$. Move the disk to tower B. Move the disk to tower C. For a_2: move the small disk to tower C (2 moves). Move the big disk to tower B (one move). Move the small disk back to tower A (two moves). Move the big disk to tower C (one move). Move the small disk back to tower C (2 moves). So, $\boxed{a_2 = 8}$. For a_3: move the two small disks to Tower C (a_2 moves). Move the big disk to tower B (1 move). Move the two small disks to tower A (a_2 moves). Move the big disk to tower C (1 move). Move the two small disks back to tower C (a_2 moves). So, $a_3 = 3a_2 + 2 = \boxed{26}$. Of course, we now see how to do the general problem. Move the $n-1$ small disks to Tower C (a_{n-1} moves). Move the big disk to tower B (1 move). Move the $n-1$ small disks to tower A (a_{n-1} moves). Move the big disk to tower C (1 move). Move the $n-1$ small disks back to tower C (a_{n-1} moves). So, $\boxed{a_n = 3a_{n-1} + 2}$.

6. **Does there exist a one-to-one and onto function from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers? Explain.**

View the elements of $\mathbb{N} \times \mathbb{N}$ as sitting in a grid:

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \dots \\
(2,1) & (2,2) & (2,3) & (2,4) & \dots \\
(3,1) & (3,2) & (3,3) & (3,4) & \dots \\
(4,1) & (4,2) & (4,3) & (4,4) & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

Notice that every element of $\mathbb{N} \times \mathbb{N}$ sits on this grid exactly once. Count off the elements of $\mathbb{N} \times \mathbb{N}$ by marching down the diagonal lines from NorthEast to SouthWest as indicated:

$$\begin{array}{ccccc}
\underset{1}{(1,1)} & \underset{2}{(1,2)} & \underset{4}{(1,3)} & \underset{7}{(1,4)} & \dots \\
\underset{3}{(2,1)} & \underset{5}{(2,2)} & \underset{8}{(2,3)} & (2,4) & \dots \\
\underset{6}{(3,1)} & \underset{9}{(3,2)} & (3,3) & (3,4) & \dots \\
\underset{10}{(4,1)} & (4,2) & (4,3) & (4,4) & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

7. **Let r, m, and n be integers with $0 \le r \le m, n$. Simplify $\binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \cdots + \binom{n}{r-1}\binom{m}{1} + \binom{n}{r}\binom{m}{0}$. (Your answer should not have any $\dots$ or any summation signs.)**

$$\boxed{\binom{n+m}{r}}$$

Suppose a committee has n men and m women. How many subcommittees of size r can be formed?

8. **Let n be a positive integer. Simplify $\sum_{k=0}^{n} 2^k \binom{n}{k}$. (Your answer should not have any $\ldots$ or any summation signs.)**

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = (2+1)^n = \boxed{3^n}$$

9. **How many four tuples (i, j, k, ℓ) are there with $3 \le i \le j \le k \le \ell \le 10$.**

Think of 8 bins. The first bin contains 3's. The second bin contains 4's. The eighth bin contains 10's. I walk past the bins and pick out 4 numbers. There is a one-to-one correspondence between legal 4-tuples and work orders which consist of 4 picks and 7 switches. The answer is $\boxed{\binom{4+7}{4}}$.

10. **How many bit strings of length 8 contain exactly three 1's? (A bit string is a string of 0's and 1's.)**

$\boxed{\binom{8}{3}}$.

11. **How many integers between 1 and 1000 are relatively prime to 28?**

Notice that $4(250) = 1000$, $7(142) = 994$, and $28(35) = 980$. There are 1000 integers between 1 and 1000; 250 of these integers are divisible by 4; 142 are divisible by 7. We have pitched out 35 twice. We must compensate for our over eagerness.

$$\boxed{1000 - 250 - 142 + 35}$$

12. **A group of eight people attend a movie together. John and Mary are part of the group and they refuse to sit next to one another. How many ways may the eight people be arranged in a row?**

There are $8!$ ways for the people to sit; $7!$ of these have John on Mary's left; $7!$ of these have John on Mary's right side.

$$\boxed{8! - 2(7!)}$$

13. **If the largest of 87 consecutive integers is 326, what is the smallest?**

$$\boxed{326 - 87 + 1}$$

14. **Let $A = \{t, u, v, w\}$ and let S_1 be the set of all subsets of A that do not contain w and S_2 the set of all subsets of A that do contain w.**
   - **(a) Find S_1. The elements of S_1 are: $\emptyset$, $\{t\}$, $\{u\}$, $\{v\}$, $\{t,u\}$, $\{t,v\}$, $\{u,v\}$, $\{t,u,v\}$.**
   - **(b) Find S_2. The elements of S_2 are: $\{w\}$, $\{t,w\}$, $\{u,w\}$, $\{v,w\}$, $\{t,u,w\}$, $\{t,v,w\}$, $\{u,v,w\}$, $\{t,u,v,w\}$.**

15. **Let $A = \{1,2,3\}$ and $B = \{4,5\}$. List the elements of $A \times B$.**

The elements of $A \times B$ are $(1,4)$, $(1,5)$, $(2,4)$, $(2,5)$, $(3,4)$, $(3,5)$.

16. **If A, B, and C are sets, is $A \cup (B \cap C) = (A \cup B) \cap C$? Prove or give a counterexample.**

False. Take $A = \{1\}$, $B = C = \emptyset$. The left side is $\{1\}$. The right side is $\emptyset$.

17. **Prove $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.**

Proof by induction. The base case is okay. If $n = 1$, then the left side of the proposed equation is 1 and the right side is $\frac{1(2)(3)}{6}$, which is also 1. The induction hypothesis is that for some fixed n

(IH)
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

We show that the formula holds at $n+1$. That is we show that

(WMS)
$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

The left side of (WMS) is

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2,$$

and, by (IH), this is

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = (n+1)\left[\frac{n(2n+1)}{6} + \frac{6(n+1)}{6}\right]$$

$$= (n+1)\left[\frac{2n^2+n+6n+6}{6}\right] = (n+1)\left[\frac{2n^2+7n+6}{6}\right] = (n+1)\left[\frac{(2n+3)(n+2)}{6}\right],$$

which is the right side of (WMS).

18. **Consider the infinite sequence** $a_1 = \frac{1}{4}$**,** $a_2 = \frac{2}{9}$**,** $a_3 = \frac{3}{16}$**,** $a_4 = \frac{4}{25}$**,** $a_5 = \frac{5}{36}$**,** $a_6 = \frac{6}{49}$**,** $\ldots$ **. What is a formula for** a_n**?**

$$\boxed{a_n = \frac{n}{(n+1)^2}}$$

19. **Prove that there are infinitely many prime integers.**

Suppose there are only finitely many prime integers. Call these integers $p_1, p_2, \ldots, p_s$. Consider $N = p_1 \times p_2 \times \cdots \times p_s + 1$. We see that N is bigger than all of the prime integers. We also see that N is not divisible by any of the primes $p_1, p_2, \ldots, p_s$. Since N is not divisible by any prime, N must be prime; but N is too big to be prime. This is a contradiction. Our supposition must be false. There are an infinite number of prime integers.

20. **True or False. The sum of two irrational numbers is irrational. Give a proof or a counterexample.**

False. $2 - \sqrt{2}$ and $\sqrt{2}$ are both irrational, but the sum of these two numbers is 2 which is rational.

21. **Prove that the square of any integer has the form** $3k$ **or** $3k+1$ **for some integer** k**.**

Let n be an arbitrary integer. There are three cases: $n = 3\ell$, $n = 3\ell + 1$, or $n = 3\ell + 2$ for some integer ℓ.

In the first case, $n = 3\ell$, so $n^2 = 9\ell^2 = 3(3\ell^2)$, which has the form $3k$, with $k = 3\ell^2$.

In the second case, $n = 3\ell + 1$, so $n^2 = 9\ell^2 + 6\ell + 1 = 3(3\ell^2 + 2\ell) + 1$, which has the form $3k+1$, with $k = 3\ell^2 + 2\ell$.

In the third case, $n = 3\ell + 2$, so $n^2 = 9\ell^2 + 12\ell + 4 = 3(3\ell^2 + 4\ell + 1) + 1$, which has the form $3k+1$, with $k = 3\ell^2 + 4\ell + 1$.

22. **True or False. If** a**,** b**, and** c **are integers with** $a|bc$**, then** $a|b$ **or** $a|c$**. Give a proof or a counterexample.**

False. $6 | 2 \cdot 3$, but $6 \nmid 2$ and $6 \nmid 3$.

23. **Write the repeating decimal** $6.2\overline{34}$ **as the ratio of two integers.**

Let $d = 6.2\overline{34}$. Observe that $100d = 623.4\overline{34}$, so $99d = 100d - d = 623.4\overline{34} - 6.2\overline{34} = 617.2$. Therefore $d = \frac{617.2}{99} = \boxed{\frac{6172}{990}}$.

24. **Write "Being divisible by** 6 **is a sufficient condition for being divisible by** 3**." in if-then form.**

If a number is divisible by 6, then the number is divisible by 3.

25. **Prove that** $p \to q$ **is logically equivalent to the contrapositive of** $p \to q$ **.**

The contrapositive of $p \to q$ is $\sim q \to \sim p$. These statements take the same truth value as the truth values of p and q are T and F:

p	q	$p \to q$	$\sim q \to \sim p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T