Math 174  Fall 2003  Final Exam.

PRINT Your Name: __________________________

There are 25 problems on 7 pages. Each problem is worth 4 points. The exam is worth 100 points. \[\text{CIRCLE} \] your answers. No Calculators.

WHEN YOU DO SOMETHING CLEVER, EXPLAIN YOUR WORK.

If I know your e-mail address, I will e-mail your course grade to you. If I don’t already know your e-mail address and you want me to know it, then send me an e-mail. Otherwise, get your course grade from VIP.

I will post the solutions on my website later today.

You may leave the binomial coefficient \( \binom{n}{r} \) in any of your answers.

1. Consider the relation \( \leq \) on the set of real numbers. Is this relation reflexive, symmetric, transitive? Explain.

2. If \( a \) and \( b \) are integers, then \( a \equiv b \pmod{5} \) if and only if \( 5 | (a - b) \). Describe the equivalence classes that correspond to this equivalence relation.

3. Suppose \( n \equiv 1 \pmod{5} \). Give a formula for \( \lfloor \frac{n}{5} \rfloor \) which does not involve \( \lfloor \rfloor \).

4. Solve the recurrence relation \( a_0 = 1000, \ a_n = (1.05)a_{n-1} + 100 \) for \( 1 \leq n \).

5. Consider the Tower of Hanoi problem. There are three towers in a row: tower A, tower B, and tower C. There are \( n \) disks of different sizes stacked on tower A. One must move all \( n \) disks to tower C. One may NEVER place a bigger disk on top of a smaller disk. In the present problem, one may move a disk only to an ADJACENT tower. Let \( a_n \) be the minimum number of moves needed to transfer a stack of \( n \) disks from tower A to tower C. Find \( a_1, a_2, a_3 \). Find a recurrence relation for \( a_1, a_2, a_3, \ldots \).

6. Does there exist a one-to-one and onto function from \( \mathbb{N} \) to \( \mathbb{N} \times \mathbb{N} \), where \( \mathbb{N} \) is the set of positive integers? Explain.

7. Let \( r, m, \) and \( n \) be integers with \( 0 \leq r \leq m, n \). Simplify \( \binom{n}{0} \binom{m}{r} + \binom{n}{1} \binom{m}{r-1} + \binom{n}{2} \binom{m}{r-2} + \cdots + \binom{n}{r-1} \binom{m}{1} + \binom{n}{r} \binom{m}{0} \). (Your answer should not have any \( \ldots \) or any summation signs.)

8. Let \( n \) be a positive integer. Simplify \( \sum_{k=0}^{n} 2^k \binom{n}{k} \). (Your answer should not have any \( \ldots \) or any summation signs.)

9. How many four tuples \( (i, j, k, \ell) \) are there with \( 3 \leq i \leq j \leq k \leq \ell \leq 10 \).

10. How many bit strings of length 8 contain exactly three 1’s? (A bit string is a string of 0’s and 1’s.)
11. How many integers between 1 and 1000 are relatively prime to 28?

12. A group of eight people attend a movie together. John and Mary are part of the group and they refuse to sit next to one another. How many ways may the eight people be arranged in a row?

13. If the largest of 87 consecutive integers is 326, what is the smallest?

14. Let \( A = \{t, u, v, w\} \) and let \( S_1 \) be the set of all subsets of \( A \) that do not contain \( w \) and \( S_2 \) the set of all subsets of \( A \) that do contain \( w \).
   (a) Find \( S_1 \).
   (b) Find \( S_2 \).

15. Let \( A = \{1, 2, 3\} \) and \( B = \{4, 5\} \). List the elements of \( A \times B \).

16. If \( A \), \( B \), and \( C \) are sets, is \( A \cup (B \cap C) = (A \cup B) \cap C \)? Prove or give a counterexample.

17. Prove \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \).

18. Consider the infinite sequence \( a_1 = \frac{1}{4} \), \( a_2 = \frac{2}{9} \), \( a_3 = \frac{3}{16} \), \( a_4 = \frac{4}{25} \), \( a_5 = \frac{5}{36} \), \( a_6 = \frac{6}{49} \), \( \ldots \). What is a formula for \( a_n \)?

19. Prove that there are infinitely many prime integers.

20. True or False. The sum of two irrational numbers is irrational. Give a proof or a counterexample.

21. Prove that the square of any integer has the form \( 3k \) or \( 3k+1 \) for some integer \( k \).

22. True or False. If \( a \), \( b \), and \( c \) are integers with \( a \mid bc \), then \( a \mid b \) or \( a \mid c \). Give a proof or a counterexample.

23. Write the repeating decimal \( 6.2\overline{34} \) as the ratio of two integers.

24. Write “Being divisible by 6 is a sufficient condition for being divisible by 3.” in if-then form.

25. Prove that \( p \rightarrow q \) is logically equivalent to the contrapositive of \( p \rightarrow q \).