Math 174, Exam 4, Fall 2003, Solutions

PRINT Your Name:________________________________

There are 10 problems on 4 pages. Each problem is worth 5 points. The exam is worth 50 points.

$\boxed{CIRCLE}$ your answers. **No Calculators.**

WHEN YOU DO SOMETHING CLEVER, EXPLAIN YOUR WORK.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office door later today, you may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 12:30 today.

You may leave the binomial coefficient $\binom{n}{r}$ in any of your answers.

1. **Flip a coin ten times, what is the probabilty that the coin lands "Heads" exactly three times.**

The sample space (all words of length 10 made out of "H" and "T") has size 2^{10}. The number of words which consist of 3 "H" and 7 "T" is $\binom{10}{3}$. The answer is $\boxed{\frac{\binom{10}{3}}{2^{10}}}$.

2. **How many license plates are possible if every license plate consists of three letters followed by three numerical digits and no letter or digit is repeated.**

$\boxed{26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8}$.

3. **How many solutions does the equation $y_1 + y_2 + y_3 + y_4 = 32$ have, if every y_i is an integer at least 5?**

We solve instead $y_1' + y_2' + y_3' + y_4' = 12$ where each y_i' is a nonnegative integer and $y_i = y_i' + 5$. We picture four bins (labeled y_1', y_2', y_3', and y_4') in a row. We have 12 rocks to distribute among the four bins. Each such distribution gives rise to a solution to the original problem. We count the number of work orders consisting of 12 drops and 3 switches to count the number of distributions. The answer is $\boxed{\binom{15}{3}}$.

4. **If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions and $g \circ f\colon X \to Z$ is one-to-one, must f and g both be one-to-one? Prove or give a counterexample.**

The function g does not have to be one-to-one. Consider $X = Z = \{1\}$, $Y = \{1, 2\}$, $f(1) = 1$, and $g(1) = g(2) = 1$. We see that $g \circ f\colon X \to Z$ is one-to-one, but $g(1) = g(2)$ with $1 \neq 2$, so g is not one-to-one.

5. **Find the coefficient of x^7 in $(2x+3)^{10}$.**

The term x^7 occurs in $\binom{10}{7}(2x)^7 3^3$. The coefficient of x^7 is $\boxed{\binom{10}{7}2^7 3^3.}$

6. **Given an example of a function from $\mathbb{Z}$ to $\mathbb{Z}$ which is onto, but which is not one-to-one.**

Consider $f:\mathbb{Z}\to\mathbb{Z}$ which is given by

$$f(n)=\begin{cases} n & \text{if } n\le 0 \\ n-1 & \text{if } 1\le n. \end{cases}$$

We see that $f(1)=f(0)=0$, so f is not one-to-one. On the other hand, it is clear that f is onto.

7. **Simplify $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}$. (Your answer should not have any $\dots$ or any summation signs.)**

This expression is equal to $\boxed{2^n}$. There are two ways to think of the problem. One could apply the binomial theorem

$$2^n=(1+1)^n=\sum_{k=0}^{n}1^k1^{n-k}\binom{n}{k}=\sum_{k=0}^{n}\binom{n}{k}.$$

Or one could count the number of subsets in a set of size n. There are 2^n such subsets. Also, there are $\binom{n}{0}$ subsets of size 0; $\binom{n}{1}$ subsets of size 1; $\binom{n}{2}$ subsets of size 2; $\dots$; $\binom{n}{n}$ subsets of size n.

8. **Solve the recurrence relation $a_0=3$ and $a_k=a_{k-1}+k$ for $1\le k$.**

$a_1=3+1$, $a_2=3+1+2$, $a_3=3+1+2+3$. Soon, we recoginize the pattern, $a_n=3+(1+2+\cdots+n)$. So, $\boxed{a_n=3+\frac{n(n+1)}{2}}$. We can check this answer. When $n=0$ my formula gives 3. Also, if we assume by induction, that $a_n=3+\frac{n(n+1)}{2}$, then

$$\begin{aligned} a_{n+1}=a_n+(n+1)=3+\frac{n(n+1)}{2}+(n+1)&=3+\frac{n(n+1)+2(n+1)}{2} \\ &=3+\frac{(n+1)(n+2)}{2}, \end{aligned}$$

as desired.

9. **A person makes an initial deposit of \$1,000 to a bank account earning interest at a rate of 6% per year compounded monthly (so the interest earned each month is $\frac{.06}{12}=.005$), and each month she adds an additional \$100 to the account. For each nonnegative integer n, let A_n be the amount in the account at the end of n months. Find a recurrence relation relating A_n to A_{n-1}.**

$\boxed{A_n=1.005A_{n-1}+100}$

10. **A single pair of rabbits (male and female) is born at the beginning of a year. Assume:**
    (1) Rabbit pairs are not fertile during the first month of life, but there after give birth to four new male/female pairs at the end of every month;
    (2) No rabbits die.

Let r_n equal the number of pairs of rabbits alive at the end of month n. Start with $r_0 = 1$. Find r_1, r_2 and r_3. Find a recurrence relation relating r_n to earlier r_k's.

$$r_1 = 1, \quad r_2 = 1 + 4 = 5, \quad r_3 = 5 + 4 = 9, \quad r_n = r_{n-1} + 4r_{n-2}.$$