Math 142, Final exam, Spring, 2004 Solutions

PRINT Your Name:____________________________________

There are 20 problems on 10 pages. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**. Otherwise, get your grade from VIP.

I will post the solutions on my website when the exam is finished.

1. **Find** $\int \sin^2 x \cos^3 x \, dx$ **. Check your answer.**

We see that

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx.$$

Let $u = \sin x$, so $du = \cos x$. It follows that the integral is

$$\int u^2(1-u^2)\,du = \int (u^2 - u^4)\,du = \frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.}$$

Check. The derivative of the proposed answer is

$$\sin^2 x \cos x - \sin^4 x \cos x = \sin^2 \cos x (1 - \sin^2 x) \checkmark.$$

2. **Find** $\int_{-3}^{0} \frac{1}{(x+1)^2}\,dx$ **. Check your answer.**

The function $x \mapsto \frac{1}{(x+1)^2}$ becomes infinite at $x = -1$. This integral is improper. (Notice that the function $x \mapsto \frac{1}{(x+1)^2}$ is positive wherever it is defined. See the picture. If the integral has a finite answer, then the answer is the area of a region; hence, must be positive.) At any rate,

$$\int_{-3}^{0} \frac{1}{(x+1)^2}\,dx = \lim_{b \to -1^-} \int_{-3}^{b} \frac{1}{(x+1)^2}\,dx + \lim_{a \to -1^+} \int_{a}^{0} \frac{1}{(x+1)^2}\,dx$$

$$= \lim_{b \to -1^-} \left.\frac{-1}{x+1}\right|_{-3}^{b} + \lim_{a \to -1^+} \left.\frac{-1}{x+1}\right|_{a}^{0}$$

$$= \lim_{b \to -1^-} \frac{-1}{b+1} + \frac{1}{-2} + \lim_{a \to -1^+} \frac{-1}{1} - \frac{-1}{a+1} = +\infty - \frac{1}{2} - 1 + \infty = \infty.$$

$\boxed{\text{The integral diverges.}}$ The answer " $-\frac{3}{2}$ " is worth zero.

3. **Find** $\int x \sin x \, dx$ **. Check your answer.**

Let $u = x$ and $dv = \sin x \, dx$. It follows that $du = dx$, $v = -\cos x$, and

$$\int x \sin x \, dx = \int u \, dv = uv - \int v \, du = -x \cos x + \int \cos x \, dx =$$

$$\boxed{-x \cos x + \sin x + C}.$$

Check. The derivative of the proposed answer is $-x(-\sin x) - \cos x + \cos x$. ✓

4. **Find** $\int x \sin(x^2) \, dx$ **. Check your answer.**

Let $u = x^2$. It follows that $du = 2x \, dx$ and the integral is

$$(1/2) \int \sin u \, du = -(1/2) \cos u + C = \boxed{-(1/2) \cos(x^2) + C}.$$

Check. The derivative of the proposed answer is $-(1/2)(-\sin(x^2))2x$. ✓

5. **Find the area of the region bounded by** $y = \ln x$ **, the** x **-axis, and** $x = 2$ **.**

Look at the picture. The area is $\int_1^2 \ln x \, dx$. Let $u = \ln x$ and $dv = dx$. It follows that $du = 1/x \, dx$, $v = x$, and the area is

$$\left(uv - \int v \, du\right)\Big|_1^2 = \left(x \ln x - \int dx\right)\Big|_1^2 = (x \ln x - x)|_1^2 = 2 \ln 2 - 2 - (1 \ln 1 - 1)$$

$$= \boxed{2 \ln 2 - 1}.$$

By the way, we see that

$$\frac{d}{dx}(x \ln x - x) = x\frac{1}{x} + \ln x - 1 = \ln x,$$

as desired.

6. **Let** $f(x) = e^{-x^2}$ **. Where is** $f(x)$ **increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of** $y = f(x)$ **. Find the horizontal asymptotes of** $y = f(x)$ **. Graph** $y = f(x)$ **.**

We see that $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x) = 0$. We conclude that the x -axis is a horizontal asymptote of $y = f(x)$. We calculate that $f'(x) = -2xe^{-x^2}$. It is clear that $f'(x)$ is zero only when $x = 0$. Furthermore, $f'(x)$ is positive when $x < 0$ and $f'(x)$ is negative, when $0 < x$. In other words, $\boxed{f(x) \text{ is increasing for } x < 0}$; $\boxed{f(x) \text{ is decreasing for } 0 < x}$; and

$\boxed{(0,1) \text{ is the local maximum point of the graph}}$. There are no local minimum points. We see that

$$f''(x) = -2x(-2x)e^{-x^2} - 2e^{-x^2} = 2e^{-x^2}(2x^2-1).$$

The factor $2e^{-x^2}$ is always positive. So the sign of $f''(x)$ is the same as the sign of $(2x^2-1) = (\sqrt{2}x-1)(\sqrt{2}x+1)$. Thus, $f''(x)$ is zero when $x = \pm\frac{1}{\sqrt{2}}$; $f''(x)$ is positive for $\frac{1}{\sqrt{2}} < x$, also for $x < -\frac{1}{\sqrt{2}}$; and $f''(x)$ is negative for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. That is,

$$\boxed{\begin{array}{c}\text{the graph is concave up for for } \frac{1}{\sqrt{2}} < x, \text{ also for } x < -\frac{1}{\sqrt{2}} \\ \text{the graph is concave down for } -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}, \text{ and} \\ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}) \text{ and } (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}) \text{ are the inflection points of the graph.}\end{array}}$$

7. **Find** $\lim\limits_{x\to 0} \frac{\sin(x^2)-x^2+\frac{x^6}{6}}{x^{10}}$ **. Justify your answer.**

We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots .$$

Thus,

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots ;$$

hence,

$$\sin(x^2) - x^2 + \frac{x^6}{6} = +\frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots = x^{10}\left[+\frac{1}{5!} - \frac{x^4}{7!} + \dots\right];$$

We conclude that

$$\lim_{x\to 0} \frac{\sin(x^2) - x^2 + \frac{x^6}{6}}{x^{10}} = \lim_{x\to 0}\left[+\frac{1}{5!} - \frac{x^4}{7!} + \dots\right] = \boxed{\frac{1}{5!}}.$$

8. **Find** $\int \frac{dx}{\sqrt{x^2-1}}$ **. Check your answer.**

Let $x = \sec\theta$. It follows that $dx = \sec\theta\tan\theta\, d\theta$. Observe that $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$. Thus,

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|$$

$$= \boxed{\ln|x + \sqrt{x^2-1}| + C}.$$

Check. The derivative of the proposed answer is

$$\frac{1 + \frac{2x}{2\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} = \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}(x+\sqrt{x^2-1})}. \checkmark$$

9. **Find** $\int \frac{3x^2+4x+4}{x^3+4x}\,dx$ **. Check your answer.**

Multiply both sides of

$$\frac{3x^2+4x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

by $x(x^2+4)$ to get

$$3x^2+4x+4 = A(x^2+4) + (Bx+C)x = (A+B)x^2 + Cx + 4A.$$

Equate the corresponding coefficients to see

$$3 = A+B, \quad 4 = C, \quad \text{and} \quad A = 1.$$

Thus, $B = 2$. By the way

$$\frac{1}{x} + \frac{2x+4}{x^2+4} = \frac{x^2+4+2x^2+4x}{x^3+4x} = \frac{3x^2+4x+4}{x^3+4x},$$

as expected. So,

$$\int \frac{3x^2+4x+4}{x^3+4x}\,dx = \int \frac{1}{x} + \frac{2x+4}{x^2+4}\,dx = \boxed{\ln|x| + \ln(x^2+4) + 2\arctan\left(\frac{x}{2}\right) + C}.$$

Check. The derivative of the proposed answer is

$$\frac{1}{x} + \frac{2x}{x^2+4} + 2\frac{\frac{1}{2}}{\left(\frac{x}{2}\right)^2+1} = \frac{1}{x} + \frac{2x}{x^2+4} + \frac{1}{\frac{x^2}{4}+1} = \frac{1}{x} + \frac{2x+4}{x^2+4}. \checkmark$$

10. **Where does the function**

$$f(x) = \sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n \cdot n^2}$$

converge? Justify your answer.

Use the ratio test. Let

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{|x+3|^{n+1}}{2^{n+1}\cdot(n+1)^2}}{\frac{|x+3|^n}{2^n\cdot n^2}} = \lim_{n\to\infty} \frac{|x+3|n^2}{2(n+1)^2} = \lim_{n\to\infty} \frac{|x+3|}{2(1+\frac{1}{n})^2}$$
$$= \frac{|x+3|}{2}.$$

If $\rho < 1$, then $f(x)$ converges. If $1 < \rho$, then $f(x)$ diverges. We see that $\rho < 1$ when $\frac{|x+3|}{2} < 1$; that is, $|x+3| < 2$; or $-2 < x+3 < 2$. So, if $-5 < x < -1$, then $f(x)$ converges. If $x < -5$ or $-1 < x$, then $f(x)$ diverges. Now we study the endpoints. At $x = -1$, $f(-1) = \sum\limits_{n=1}^{\infty} \frac{2^n}{2^n\cdot n^2} = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$. This is the p -series with $p = 2$. This series converges. At $x = -5$, $f(-5) = \sum\limits_{n=1}^{\infty} \frac{(-1)^n 2^n}{2^n\cdot n^2} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. We have already noticed that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$. Thus, the Absolute Convergence Test tells us that $f(-5)$ also converges. We conclude that

$$\boxed{f(x) \text{ converges for } -5 \le x \le -1 \text{ and } f(x) \text{ diverges everywhere else.}}$$

11. **What familiar function is equal to**

$$f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \ldots ?$$

Justify your answer.

We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$
$$= x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \ldots\right).$$

Thus

$$\boxed{f(x) = \frac{\sin x}{x}.}$$

12. **Find the limit of the sequence whose** n^{th} **term is** $a_n = \left(1 - \frac{1}{3n}\right)^n$**.**

Justify your answer.

We know that $\lim_{n\to\infty}(1 + \frac{r}{n})^n = e^r$. It follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{\frac{-1}{3}}{n}\right)^n = \boxed{e^{\frac{-1}{3}}}.$$

13. **Find** $\frac{d}{dx}(x2^x)$**.**

We see that

$$\frac{d}{dx}(x2^x) = \boxed{x2^x \ln 2 + 2^x.}$$

14. **Does the series** $\sum_{n=1}^{\infty} \frac{n}{e^n}$ **converge or diverge? Justify your answer.**

I use the ratio test. (The integral test will also work.) Let

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \lim_{n\to\infty} \frac{n+1}{ne} = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{e} = \frac{1}{e} < 1.$$

We conclude that $\boxed{\sum_{n=1}^{\infty} \frac{n}{e^n} \text{ converges.}}$

15. **Approximate** $e^{\frac{-1}{10}}$ **with an error at most** $\frac{1}{1000}$ **. Justify your answer.**

We know that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

for all x . It follows that

$$e^{\frac{-1}{10}} = 1 + \frac{-1}{10} + \frac{(\frac{-1}{10})^2}{2} + \frac{(\frac{-1}{10})^3}{3!} + \dots ;$$

that is,

$$e^{\frac{-1}{10}} = 1 - \frac{1}{10} + \frac{1}{2(10)^2} - \frac{1}{3!(10)^3} + \dots .$$

Apply the alternating series test. The nearest series alternates. The terms go to zero. The absolute value of the terms are decreasing. The alternating series test tells us that the series converges and that

$$|e^{\frac{-1}{10}} - s_n| \le a_{n+1},$$

where s_n is the n^{th} partial sum. In particular

$$\left| e^{\frac{-1}{10}} - \left(1 - \frac{1}{10} + \frac{1}{2(10)^2}\right)\right| \le \frac{1}{3!(10)^3} < \frac{1}{1000}.$$

Thus,

$$\boxed{1 - \frac{1}{10} + \frac{1}{200} \text{ approximates } e^{\frac{-1}{10}} \text{ with an error at most } \frac{1}{1000}.}$$

16. **Find the Taylor polynomial** $P_3(x)$ **for** $f(x) = \ln x$ **about** $c = 1$ **.**

We know that

$$\begin{array}{ll} f(x) = \ln x & f(1) = 0 \\ f'(x) = \frac{1}{x} & f'(1) = 1 \\ f''(x) = \frac{-1}{x^2} & f''(1) = -1 \\ f'''(x) = \frac{2}{x^3} & f'''(1) = +2 \\ f^{(4)}(x) = \frac{6}{x^4}. & \end{array}$$

We also know that

$$P_3(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2} + f'''(c)\frac{(x-c)^3}{3!};$$

hence,

$$\boxed{P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.}$$

17. **Take $P_3(x)$ and $f(x)$ from problem 16. Estimate the error that is introduced if $f(x)$ is approximated by $P_3(x)$ for $.9 \le x \le 1.1$. Justify your answer.**

Taylor's Theorem tells us that

$$|f(x) - P_3(x)| = |R_3(x)| = \left| \frac{f^{(4)}(z)}{4!}(x-c)^4 \right| = \frac{6|x-1|^4}{4!z^4},$$

for some z between x and 1. In our problem $-.1 \le x - 1 \le .1$; so $|x-1| \le .1$. Also, if z is between x and 1 and x is between $.9$ and 1.1, then z is between $.9$ and 1.1. In particular, since $.9 \le z$, we know that $\frac{1}{z} \le \frac{1}{.9}$. Thus,

$$|f(x) - P_3(x)| \le \frac{6(.1)^4}{4!(.9)^4} = \frac{1}{4(9^4)}.$$

We conclude that

$$\boxed{P_3(x) \text{ approximates } f(x) \text{ with an error of at most } \frac{1}{4(9^4)}, \text{ when } .9 \le x \le 1.1.}$$

18. **Suppose that the government pumps an extra $1 billion into the economy. Assume that each business and individual saves 20% of its income and spends the rest, so that of the initial $1 billion, 80% is respent by individuals and businesses. Of that amount, 80% is spent, and so forth. What is the total increase in spending due to the government action? Justify your answer.**

The total increase in spending is:

$$10^9 + .8(10^9) + (.8)^2(10^9) + (.8)^3(10^9) + \dots .$$

This is the geometric series with initial term $a = 10^9$ and ratio $r = .8$. The ratio is between -1 and 1, so the series converges to

$$\frac{a}{1-r} = \frac{10^9}{.2} = \boxed{\$5 \times 10^9}.$$

19. **Money is invested at an annual interest rate of 4% compounded continuously. How much money should be invested today in order for the investment to be worth $100,000 twenty years from today? Justify your answer.**

Let $A(t)$ be the value of the investment at time t. We know that $A(t) = A(0)e^{.04t}$. If $A(20) = \$100,000$, then

$$100,000 = A(0)e^{(.04)20} = A(0)e^{.8}$$

and

$$\boxed{A(0) = \frac{\$100,000}{e^{.8}}}.$$

(It turns out that $A(0)$ is approximately equal to $44,932.90.)

20. **Use Trapezoidal rule, with $n=4$, to approximate $\int_0^1 \frac{1}{1+x^2}\,dx$. Recall that Trapezoidal rule says that**

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]+E_n,$$

for $h=\frac{b-a}{n}$, $x_i=a+hi$, and $E_n=-\frac{(b-a)^3}{12n^2}f''(c)$ for some c with $a\le c\le b$. (Just record the sum. You are not required to add the fractions. You are not required to do anything with E_n.)

For us, $a=0$, $b=1$, $h=\frac{1}{4}$, $x_0=0$, $x_1=\frac{1}{4}$, $x_2=\frac{2}{4}=\frac{1}{2}$, $x_3=\frac{3}{4}$, and $x_4=\frac{4}{4}=1$. Our function is $f(x)=\frac{1}{1+x^2}$. Trapezoidal rule tells us that

$$\int_0^1 \frac{1}{1+x^2}\,dx \cong \frac{h}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\right].$$

Thus,

$$\boxed{\int_0^1 \frac{1}{1+x^2}\,dx \cong \frac{1}{8}\left[\frac{1}{1+0^2}+\frac{2}{1+(\frac{1}{4})^2}+\frac{2}{1+(\frac{1}{2})^2}+\frac{2}{1+(\frac{3}{4})^2}+\frac{1}{1+1^2}\right].}$$