Math 142, Spring 2004, Exam 4, Solutions

PRINT Your Name:____________________________________

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **NO CALCULATORS!**

I won't grade your exam until Monday. So don't be surprised if I don't e-mail your grade to you until then.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

I will post the solutions on my website on **Monday**.

1. **A ball is dropped from the height of 30 feet. Each time it hits the floor it rebounds to $\frac{5}{6}$ its previous height. Find the total distance it travels. Explain what you are doing.**

The ball goes down 30 feet, up $(5/6)30$ feet, down $(5/6)30$ feet, up $(5/6)^2 30$ feet, down $(5/6)^2 30$ feet, up $(5/6)^3 30$ feet, down $(5/6)^3 30$ feet, etc. The total distance travelled is

$$30+\left((5/6)60+(5/6)^2 60+(5/6)^3 60+(5/6)^4 60+\dots\right) \text{ feet.}$$

The sum inside the big parentheses is the geometric series with initial term $a=(5/6)60$ and ratio $r=5/6$. We see that $-1<r<1$, so the geometric series converges to $\frac{a}{1-r}$. Thus the total distance travelled is

$$\boxed{30+\frac{(5/6)60}{1-5/6} \text{ feet.}} = 330 \text{ feet.}$$

2. **Give a closed formula for $s_n=\sum\limits_{k=2}^{n} \frac{1}{k-1}-\frac{1}{k+1}$. (Your formula should be exactly equal to the sum I have given. Your formula should not contain any dots or any summation signs.) Explain what you are doing.**

This sum is equal to

$$s_n=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\dots+\left(\frac{1}{n-4}-\frac{1}{n-2}\right)$$
$$+\left(\frac{1}{n-3}-\frac{1}{n-1}\right)+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

The $1/3$'s cancel. The $1/4$'s cancel. The $-1/5$ cancels with the left most $+1/5$ from the first unwritten term. The $-1/6$ cancels with the left most $1/6$ of the second unwritten term., etc. Now look at the right side. The $\frac{1}{n-1}$'s cancel. The $\frac{1}{n-2}$'s cancel. The $\frac{1}{n-3}$ cancels with the right most $-\frac{1}{n-3}$ from the last unwritten term. The $\frac{1}{n-4}$ also cancels with a term in the middle. We are left with

$$\boxed{s_n=\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}.}$$

3. Approximate $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ with an error of at most $\frac{1}{500}$. Be sure to explain what you are doing and why you are allowed to do it.

The series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ is an alternating series. The terms, in absolute value, are decreasing, since $\frac{1}{n^4} > \frac{1}{(n+1)^4}$ for all n. The terms go to zero, since $\lim\limits_{n\to\infty}\frac{1}{n^4} = 0$. The Alternating Series Test applies and tells us that the series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ converges, and for each N

$$\left|\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4} - \sum_{n=1}^{N}(-1)^{n+1}\frac{1}{n^4}\right| \le \frac{1}{(N+1)^4}.$$

We look for (the smallest) N with $\frac{1}{(N+1)^4} < \frac{1}{500}$. In other words, we want $500 < (N+1)^4$. Recall that $4^4 = 256 < 500 < 625 = 5^4$. If $N = 4$, then $\frac{1}{(N+1)^4} < \frac{1}{500}$. Thus,

$$\boxed{\frac{1}{1} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} \text{ approximates } \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4} \text{ with an error at most } \frac{1}{500}.}$$

4. Give an upper bound for the difference between $\sum\limits_{n=1}^{5}\frac{1}{n^3}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^3}$. I expect your upper bound to be relatively small and correct. Be sure to explain what you are doing and why you are allowed to do it.

The difference between $\sum\limits_{n=1}^{5}\frac{1}{n^3}$ and $\sum\limits_{n=1}^{\infty}\frac{1}{n^3}$ is $\sum\limits_{n=6}^{\infty}\frac{1}{n^3}$. Draw a picture to see that

$$\sum_{n=6}^{\infty}\frac{1}{n^3} \le \int_5^{\infty} x^{-3}\,dx = \lim_{b\to\infty}\left.\frac{-1}{2x^2}\right|_5^b = \lim_{b\to\infty} -\frac{1}{2b^2} + \frac{1}{50} = \frac{1}{50}.$$

We conclude that

$$\boxed{\text{the difference between } \sum_{n=1}^{5}\frac{1}{n^3} \text{ and } \sum_{n=1}^{\infty}\frac{1}{n^3} \text{ is at most } \frac{1}{50}.}$$

5. Does the series $\sum\limits_{n=1}^{\infty}\frac{\sin n}{n^3}$ converge or diverge? Justify your answer.

The series $\sum\limits_{n=1}^{\infty}\frac{1}{n^3}$ is the p-series with $p = 3 > 1$. This series converges. $|\sin n| \le 1$; so $\frac{|\sin n|}{n^3} \le \frac{1}{n^3}$. The comparison test tells us that $\sum\limits_{n=1}^{\infty}\frac{|\sin n|}{n^3}$ converges. Thus, the Absolute Convergence Test tells us that

$$\boxed{\text{the series } \sum_{n=1}^{\infty}\frac{\sin n}{n^3} \text{ converges.}}$$

6. **Does the series** $\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n$ **converge or diverge? Justify your answer.**

The Individual Term Test for Divergence tells us that the series

$$\boxed{\text{the series } \sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n \text{ diverges;}}$$

because $\lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-1}{n}\right)^n = \frac{1}{e} \neq 0$.

7. **Does the series** $\sum_{n=1}^{\infty} \left(\frac{-3}{4}\right)^n$ **converge or diverge? Justify your answer.**

This series is the geometric series with initial term $a = \frac{-3}{4}$ and ratio $r = \frac{-3}{4}$. We see that $-1 < r < 1$. We conclude that

$$\boxed{\text{the series } \sum_{n=1}^{\infty} \left(\frac{-3}{4}\right)^n \text{ converges to } \frac{\frac{-3}{4}}{1 - \frac{-3}{4}}.}$$

8. **Does the series** $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ **converge or diverge? Justify your answer.**

Use the ratio test. We see that

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

We conclude that

$$\boxed{\text{the series } \sum_{n=1}^{\infty} \frac{2^n}{n!} \text{ converges.}}$$

9. **Where does the function**

$$f(x) = \frac{x-1}{1} + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{4} + \dots$$

converge? Justify your answer.

Write $f(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$. Apply the ratio test. Let

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{|x-1|^{n+1}}{n+1}}{\frac{|x-1|^n}{n}} = \lim_{n\to\infty} \frac{|x-1|n}{n+1} = |x-1|.$$

If $|x-1|<1$ (i.e., $-1<x-1<1$; i.e., $0<x<2$), then $f(x)$ converges. If $1<|x-1|$ (i.e., $x<0$ or $2<x$), then $f(x)$ diverges. When $x=0$, then

$$f(0)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n},$$

which is minus the Alternating Harmonic Series. This series converges. When $x=2$, then

$$f(2)=\sum_{n=1}^{\infty}\frac{(1)^n}{n},$$

which is the Harmonic Series. This series diverges. We conclude that

$$\boxed{f(x) \text{ converges for } 0\le x<2 \text{ and } f(x) \text{ diverges everywhere else.}}$$

10. **What familiar function is equal to**

$$f(x)=\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots$$

for $-1<x<1$? Justify your answer.

We know that

$$\frac{1}{1-x}=1+x+x^2+x^3+\dots \quad \text{for } -1<x<1\,.$$

Integrate both sides to see that

$$-\ln|1-x|+C=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots \quad \text{for } -1<x<1\,.$$

Notice that when $-1<x<1$, then $0<1-x$; so, $|1-x|=1-x$. Plug in $x=0$ to evaluate C:

$$-\ln 1+C=0;$$

so, $C=0$. We conclude that

$$\boxed{\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots \text{ is equal to } -\ln(1-x) \text{ for } -1<x<1.}$$