Math 142 Exam 2 Spring 2004 Solutions

PRINT Your Name:___________________________________

There are 10 problems on 6 pages. Each problem is worth 10 points. SHOW your work. $\boxed{\mathit{CIRCLE}}$ your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office after I have graded it. (If you like, I will send you an e-mail when I am finished with it.) You may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 4:00 PM today.

1. **Find** $\int \sin^3 x \cos^4 x\, dx$ **. Check your answer.**

The integral is equal to

$$\int (1-\cos^2 x)\cos^4 x \sin x\, dx.$$

Let $u = \cos x$. It follows that $du = -\sin x\, dx$ and the integral is

$$-\int (1-u^2)u^4\, du = -\int (u^4 - u^6)\, du = -\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + C$$

$$= \boxed{-\left(\frac{\cos^5 x}{5} - \frac{\cos^7 x}{7}\right) + C.}$$

Check: The derivative of the proposed answer is

$$-\big(\cos^4 x(-\sin x) - \cos^6 x(-\sin x)\big) = \sin x \cos^4 x(1-\cos^2 x).\checkmark$$

2. **Find** $\int \frac{x}{x^2+4}\, dx$ **. Check your answer.**

Let $u = x^2+4$. It follows that $du = 2x\, dx$ and the integral is equal to

$$\tfrac{1}{2}\int \tfrac{du}{u} = \tfrac{1}{2}\ln|u| + C = \boxed{\tfrac{1}{2}\ln|x^2+4| + C.}$$

Check: The derivative of the proposed answer is $\frac{1}{2}\frac{2x}{x^2+4}$. ✓

3. Find $\int \frac{1}{x^2+4}\,dx$. Check your answer.

Maybe you remember that $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$. If so, the answer is $\boxed{\frac{1}{2}\arctan\frac{x}{2}+C.}$ Maybe you would rather say that the problem is equal to

$$\int \frac{1}{4(\frac{x^2}{4}+1)}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx.$$

Let $u=\frac{x}{2}$. It follows that $du=\frac{dx}{2}$ and the integral is

$$\frac{2}{4}\int \frac{1}{u^2+1}\,du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan\frac{x}{2} + C,$$

which is what we got before. Maybe you would rather do a trig substitution. Let $x=2\tan\theta$. It follows that $dx = 2\sec^2\theta\,d\theta$,

$$x^2+4 = 4\tan^2\theta+4 = 4(\tan^2 x+1) = 4\sec^2\theta,$$

and the integral is

$$\int \frac{2\sec^2\theta}{4\sec^2\theta}d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C = \frac{1}{2}\arctan\frac{x}{2}+C.$$

Once again, we get the same answer.

Check: The derivative of the proposed answer is

$$\frac{1}{2}\frac{\frac{1}{2}}{(\frac{x}{2})^2+1} = \frac{1}{4((\frac{x}{2})^2+1)}.\checkmark$$

4. Find $\int \frac{1}{\sqrt{x^2+4}}\,dx$.

We do a Trig substitution. Let $x=2\tan\theta$. It follows that $dx=2\sec^2\theta\,d\theta$,

$$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4(\tan^2 x+1)} = \sqrt{4\sec^2\theta} = 2\sec\theta,$$

and the integral is

$$\int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C = \boxed{\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|+C.}$$

Check: The derivative of the proposed answer is

$$\frac{\frac{2x}{4\sqrt{x^2+4}}+\frac{1}{2}}{\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}} = \frac{\frac{x}{\sqrt{x^2+4}}+1}{\sqrt{x^2+4}+x} = \frac{\left(\frac{x}{\sqrt{x^2+4}}+1\right)\sqrt{x^2+4}}{\left(\sqrt{x^2+4}+x\right)\sqrt{x^2+4}} = \frac{x+\sqrt{x^2+4}}{\left(\sqrt{x^2+4}+x\right)\sqrt{x^2+4}}$$

$$= \frac{1}{\sqrt{x^2+4}}.\checkmark$$

5. **Find** $\int \cos^4 x\, dx$ **.**

Use $\cos^2 x = \frac{1}{2}(1+\cos 2x)$. The integral is

$$\int \left(\tfrac{1}{2}(1+\cos 2x)\right)^2 dx = \tfrac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \tfrac{1}{4}\int (1 + 2\cos 2x + \tfrac{1}{2}(1+\cos 4x))\, dx$$

$$= \tfrac{1}{4}\int (\tfrac{3}{2} + 2\cos 2x + \tfrac{1}{2}\cos 4x)\, dx$$

$$= \tfrac{1}{4}(\tfrac{3}{2}x + \sin 2x + \tfrac{1}{8}\sin 4x) + C.$$

6. **If** $y = \arcsin(e^{2x})$ **, then find** $\frac{dy}{dx}$ **.**

We know that

$$\frac{dy}{dx} = \boxed{\frac{2e^{2x}}{\sqrt{1-e^{4x}}}.}$$

7. **Evaluate** $\sin(2\arccos(\frac{1}{10}))$ **.**

We know that $\sin 2\theta = 2\sin\theta\cos\theta$. So

$$\sin(2\arccos(\tfrac{1}{10})) = 2\sin(\arccos(\tfrac{1}{10}))\cos(\arccos(\tfrac{1}{10})).$$

It is clear that $\cos(\arccos(\frac{1}{10})) = \frac{1}{10}$. Draw a right triangle with adjacent equal to 1 and hypotenuse equal to 10. The opposite then must be $\sqrt{99}$. We conclude that $\sin(\arccos(\frac{1}{10})) = \frac{\sqrt{99}}{10}$ and the answer is

$$2\frac{\sqrt{99}}{10}\frac{1}{10} = \boxed{\frac{\sqrt{99}}{50}.}$$

8. **Find the area of the region bounded by** $y = e^x$ **, the** $y-$ **axis, and the line** $y = e^3$ **.**

Look at the picture. The area is

$$\int_0^3 (e^3 - e^x)\, dx = (e^3 x - e^x)|_0^3 = 3e^3 - e^3 - (0-1) = \boxed{2e^3 + 1.}$$

9. **Newton's law of cooling states that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium. Thus, if an object is taken from an oven at $400°$ F and left to cool in a room at $70°$ F, then its temperature T after t hours will satisfy the differential equation**
$$\frac{dT}{dt} = k(T - 70).$$
If the temperature fell to $200°$ F after one hour, what will it be after 4 hours? (You may leave "ln" in your answer.)

We know $T(0) = 400°$ and $T(1) = 200°$. We want $T(4)$. Separate the variables in the differential equation:
$$\int \frac{dT}{T-70} = \int k\,dt.$$
Integrate both sides to get
$$\ln|T - 70| = kt + C.$$
Exponentiate to see $|T - 70| = e^C e^{kt}$ or $T - 70 = \pm e^C e^{kt}$. Let $\mathcal{C}$ be the connstant $\pm e^C$. So $T - 70 = \mathcal{C}e^{kt}$. Plug in $t = 0$ to see that $400 - 70 = \mathcal{C}$. So, $T - 70 = 330e^{kt}$. Plug in $t = 1$ to see that $200 - 70 = 330e^k$. It follows that $\frac{130}{330} = e^k$ and $\ln\frac{13}{33} = k$. Thus,
$$T(t) = 70 + 330e^{t\ln\frac{13}{33}}.$$
We conclude that
$$\boxed{T(4) = 70 + 330e^{4\ln\frac{13}{33}}}.$$
The answer is given in degrees F.

10. **Let $f(x) = xe^x$. Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of $y = f(x)$. Graph $y = f(x)$.**

We know that $f'(x) = xe^x + e^x = e^x(x+1)$. So, $f'(x) = 0$ when $x = -1$; $f'(x) > 0$ for $x > -1$; and $f'(x) < 0$ for $x < -1$. Thus

$$\boxed{\begin{array}{l} f(x) \text{ is decreasing for } x < -1; \\ f(x) \text{ is increasing for } x > -1; \\ (-1, -\frac{1}{e}) \text{ is a local minimum of } y = f(x); \text{ and} \\ y = f(x) \text{ does not have a local maximum.} \end{array}}$$

We also that $f''(x) = xe^x + e^x + e^x = e^x(x+2)$. So $f''(x) = 0$ when $x = -2$; $f''(x) < 0$ for $x < -2$; and $f''(x) > 0$ for $x > -2$. Thus,

$$\boxed{\begin{array}{l} f(x) \text{ is concave down for } x < -2; \\ f(x) \text{ is concave up for } x > -2; \\ (-2, -\frac{2}{e^2}) \text{ is a point of inflection of } y = f(x). \end{array}}$$

It would be nice to know $\lim\limits_{x\to-\infty} xe^x$. The factor x would like the answer to be $-\infty$. The factor e^x would like the answer to be 0. The form of the problem does not tell us who wins, but in fact, L'hopital's rule (section 9.1) will tell us that e^x over powers x and the limit is 0. The graph appears on a different page.