PRINT Your Name:__________________________________

There are 10 problems on 5 pages. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **NO CALCULATORS! CHECK** your answer whenever possible.

If I know your e-mail address, I will e-mail your grade to you. If I don't already know your e-mail address and you want me to know it, then **send me an e-mail**.

If you would like, I will leave your exam outside my office door tomorrow morning, you may pick it up any time between then and the next class. **Let me know if you are interested.**

I will post the solutions on my website at about 4:00 PM today.

1. **Find** $\int e^{2x+3}dx$ **. Check your answer.**

Let $u = 2x+3$. It follows that $du = 2dx$. The integral is

$$\tfrac{1}{2}\int e^u du = \tfrac{1}{2}e^u + C = \boxed{\tfrac{1}{2}e^{2x+3} + C.}$$

Check. The derivative of the proposed answer is $\frac{1}{2}2e^{2x+3}$. ✓

2. **Find** $\int xe^{2x^2+3}dx$ **. Check your answer.**

Let $u = 2x^2+3$. It follows that $du = 4xdx$. The integral is

$$\tfrac{1}{4}\int e^u du = \tfrac{1}{4}e^u + C = \boxed{\tfrac{1}{4}e^{2x^2+3} + C.}$$

Check. The derivative of the proposed answer is $\frac{1}{4}4xe^{2x^2+3}$. ✓

3. **If** $y = e^{(\frac{1}{x^3})} + \dfrac{1}{e^{(x^3)}}$ **, then find** $\frac{dy}{dx}$ **.**

We see that $y = e^{(x^{-3})} + e^{-x^3}$; therefore,

$$\boxed{\frac{dy}{dx} = -3x^{-4}e^{(x^{-3})} - 3x^2e^{-x^3}.}$$

4. **If** $y = \sin x \ln x$ **, then find** $\frac{dy}{dx}$ **.**

Use the product rule:

$$\boxed{\frac{dy}{dx} = \sin x\frac{1}{x} + \ln x \cos x.}$$

5. **Find** $\int \frac{\ln x}{x}dx$ **. Check your answer.**

Let $u = \ln x$. Then, $du = \frac{dx}{x}$ and the integral is equal to

$$\int udu = \frac{u^2}{2} + C = \boxed{\frac{(\ln x)^2}{2} + C.}$$

Check. The derivative of the proposed answer is $\frac{2(\ln x)}{2}\frac{1}{x}$. ✓

NOTICE. The functions $(\ln x)^2$ and $\ln x^2$ are very DIFFERENT!

6. **Find** $\int \frac{e^x}{\sqrt{e^x+1}}dx$ **. Check your answer.**

Let $u = e^x + 1$. Then $du = e^x dx$ and the integral is equal to

$$\int u^{-1/2}du = 2u^{1/2} + C = \boxed{2\sqrt{e^x+1} + C.}$$

Check. The derivative of the proposed answer is $2\frac{1}{2}(e^x+1)^{-1/2}e^x$. ✓

7. **Find the area of the region bounded by** $y = e^x$ **, the** $y-$ **axis, and the line** $y = e^2$ **.**

Look at the picture. The boundary on the left is $x = 0$, on the right is $x = 2$, on the top is $y = e^2$, and on the bottom is $y = e^x$. The area is

$$\int_0^2 e^2 - e^x dx = e^2x - e^x|_0^2 = (2e^2 - e^2) - (0 - 1) = \boxed{e^2 + 1}.$$

8. **Let** $f(x) = \frac{x-2}{x+3}$ **for** $x \neq -3$ **. Find** $f^{-1}(x)$ **. What is the domain of** $f^{-1}(x)$ **? Verify that** $f(f^{-1}(x)) = x$ **for all** x **in the domain of** $f^{-1}(x)$ **.**

Let $y = f^{-1}(x)$. Apply f to both sides to get $f(y) = x$. It follows that $\frac{y-2}{y+3} = x$; hence, $y - 2 = x(y + 3)$ and $y(1 - x) = 3x + 2$. Divide to see that $y = \frac{3x+2}{1-x}$. Thus $\boxed{f^{-1}(x) = \dfrac{3x+2}{1-x}}$. The domain of $f^{-1}(x)$ is all real numbers x except $x = 1$. Take a real number x with $x \neq 1$, then

$$f(f^{-1}(x)) = f\left(\frac{3x+2}{1-x}\right) = \frac{\frac{3x+2}{1-x} - 2}{\frac{3x+2}{1-x} + 3} = \frac{3x + 2 - 2(1-x)}{3x + 2 + 3(1-x)} = \frac{5x}{5} = x\checkmark.$$

9. **A bacterial population grows at a rate proportional to its size. Initially the population is** $12,000$ **and after** 6 **days the population is** $20,000$ **. How long will it take the population to triple? (You may leave "** $\ln$ **" in your answer.)**

Let $P(t)$ be the size of the population at time t days. We are told that $\frac{dP}{dt} = kP$ for some constant k . It follows that $P(t) = P(0)e^{kt}$. We are also told that $P(0) = 12,000$ and $P(6) = 20,000$. So

$$P(t) = 12,000e^{kt}$$

holds for all t . Plug in $t = 6$ to learn k :

$$20,000 = P(6) = 12,000e^{6k}.$$

Thus, $\frac{20,000}{12,000} = e^{6k}$ and $\ln\frac{5}{3} = 6k$; hence, $\frac{1}{6}\ln(\frac{5}{3}) = k$. Thus,

$$P(t) = 12,000e^{\frac{t\ln(\frac{5}{3})}{6}}$$

holds for all t. We are supposed to find t with $P(t) = 36,000$. So,

$$36,000 = P(t) = 12,000e^{\frac{t\ln(\frac{5}{3})}{6}},$$

that is, $3 = e^{\frac{t\ln(\frac{5}{3})}{6}}$ and $\ln 3 = \frac{t\ln(\frac{5}{3})}{6}$; therefore,

$$\boxed{t = \frac{6\ln 3}{\ln(\frac{5}{3})}\text{days.}}$$

10. **Let $f(x) = x\ln x$. What is the domain of $f(x)$? Where is $f(x)$ increasing, decreasing, concave up, and concave down? Find the local maxima, local minima, and points of inflection of $y = f(x)$. Graph $y = f(x)$.**

$$\boxed{\text{The domain of } f(x) \text{ is all positive real numbers } x.}$$

We see that $f'(x) = 1 + \ln x$. So, $f'(x) = 0$, when $0 = 1 + \ln x$; so $-1 = \ln x$; that is, $x = e^{-1}$.

$$\boxed{\text{When } 0 < x < e^{-1}, \text{ then } f'(x) \text{ is negative and } f \text{ is decreasing.}}$$

$$\boxed{\text{When } e^{-1} < x, \text{ then } f'(x) \text{ is positive and } f \text{ is increasing.}}$$

$$\boxed{\text{The point } (e^{-1}, -e^{-1}) \text{ is a local minimum. There is no local maximum.}}$$

The second derivative is $f''(x) = \frac{1}{x}$, which is always positive.

$$\boxed{\text{The graph is always concave up, never concave down.}}$$

$$\boxed{\text{There are no points of inflection.}}$$

It is clear that $f(1) = 0$. It is harder to compute $\lim\limits_{x\to 0^+} x\ln x$. The factor x would like the answer to be zero. The factor $\ln x$ would like the answer to be $-\infty$. In Chapter 9, you will learn L'hopital's rule which will show that x wins this battle and $\lim\limits_{x\to 0^+} x\ln x = 0$. The picture is on another page.