Consider the power series
\[ f(x) = 1 + \frac{x + 1}{2} + \frac{(x + 1)^2}{2^2} + \frac{(x + 1)^3}{2^3} + \ldots . \]

For which \( x \) does \( f(x) \) converge? Justify your answer.

**Answer:** The series is
\[ f(x) = \sum_{k=0}^{\infty} \frac{(x + 1)^k}{2^k}. \]

We apply the ratio test. Let
\[ \rho = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{(x + 1)^{k+1}}{2^{k+1} (x + 1)^k} \cdot \frac{2^k (x + 1)^k}{2^k} \right| = \lim_{k \to \infty} \frac{|x + 1|}{2}. \]

If \( \rho < 1 \), then \( f(x) \) converges. If \( 1 < \rho \), then \( f(x) \) diverges. We have to study \( \rho = 1 \) further. Well, \( \rho < 1 \) when \( \frac{|x + 1|}{2} < 1 \); that is \( |x + 1| < 2 \); that is, \(-2 < x + 1 < 2\); or \(-3 < x < 1\). We also see that \( 1 < \rho \) for \( x < -3 \) also for \( 1 < x \). The endpoints \( x = -3 \) and \( x = 1 \) need special attention. We see that
\[ f(-3) = \sum_{k=0}^{\infty} \frac{(-2)^k}{2^k} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{2^k} = \sum_{k=0}^{\infty} (-1)^k \]

which diverges by the Individual Term Test for Divergence since \( \lim_{k \to \infty} (-1)^k \) does not exist; and hence is not equal to zero. Also,
\[ f(1) = \sum_{k=0}^{\infty} \frac{2^k}{2^k} = \sum_{k=0}^{\infty} 1 \]

which diverges by the Individual Term Test for Divergence since \( \lim_{k \to \infty} 1 = 1 \), which is not zero. We conclude that

\[ f(x) \text{ converges for } -3 < x < 1 \text{ and diverges everywhere else.} \]